The two train separation problem on level track

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Abstract: We find driving strategies for two trains travelling on the same track in the same direction subject to given journey times so that an adequate separation is maintained between the trains and so that the total energy consumption is minimized. We assume the track is level.

Consider a single line rail track $[0, X]$ with signals placed along the track at points $0 = x_0 < \cdots < x_n = X$. It is a common safety requirement for trains travelling from $x = 0$ to $x = X$ that no two trains are allowed on the same segment (x_j, x_{j+1}) at the same time. We wish to find speed profiles $v_1 = v_1(x)$ for a leading train starting at time $t_{1,0} = 0$ and finishing at time $t_{1,n} = T_1$ and $v_2 = v_2(x)$ for a following train starting at time $t_{2,0} \geq 0$ and finishing at time $t_{2,n} = t_{2,0} + T_2$ such that the safety constraints are observed and so that the total energy consumption is minimized.

We solve the problem in two stages. For the first stage we consider a given set of times $0 = s_0 < \cdots < s_{n+1}$ where $s_n = T_1$ and $s_{n+1} = s_1 + T_2$. At this stage we wish to solve two problems—the leading train problem and the following train problem. That is we want to find $v_1 = v_1(x)$ so that the leading train leaves the point $x_0 = 0$ at time $t_{1,0} = 0$, passes through the points x_j at time $t_{1,j} \leq s_j$ for each $j = 1, \ldots, n-1$ and reaches point $x_n = X$ at time $t_{1,n} = T_1$ in such a way that energy consumption is minimized. We also want to find $v_2 = v_2(x)$ so that the following train leaves the point $x_0 = 0$ at time $t_{2,0} = s_1$, passes through the points x_j at time $t_{2,j} \geq s_{j+1}$ for each $j = 1, \ldots, n-1$ and reaches point $x_n = X$ at time $t_{2,n} = s_1 + T_1$ in such a way that energy consumption is minimized. By finding all sets of feasible appointed times $\{s_j\}_{j=0}^{n}$ it should be possible to choose the set of appointed times that minimizes the total energy consumed. We will consider systematic procedures for finding this best set of appointed times in another paper.

It is well known (Howlett 2000) that the optimal strategy for a train travelling on level track from $(x, t) = (0, 0)$ to $(x, t) = (X, T)$ is a *power-hold-coast-brake* strategy where the speed U at which braking begins is a uniquely determined function of the hold speed V. As the hold speed increases the journey time decreases. Thus there is a unique hold speed for each given journey time. For convenience we will call this an unconstrained strategy of optimal type. What happens to the strategy of optimal type when intermediate time constraints are imposed?

In the case of a leading train the intermediate constraints mean that the train must leave each section before some given time. If a proposed journey does not leave a particular section (x_{j-1}, x_j) before the appointed time s_j then the corresponding constraint $t_{1,j} \leq s_j$ is violated and the proposed journey is infeasible. In such cases the leading train must go faster on the first part of the journey$(0, x_j)$ in order to satisfy the active constraint $t_{1,j} \leq s_j$. Thus our intuitive idea will be that a strategy of optimal type for the leading train may use different hold speeds for different sections and as the journey progresses the hold speed will decrease.

In the case of a following train the intermediate constraints mean that the train must enter each section after some given time. If a proposed journey enters a particular section (x_{j-1}, x_j) before the appointed time s_j then the corresponding constraint $t_{2,j-1} \geq s_j$ is violated and the proposed journey is infeasible. In such cases the following train must go slower on the first part of the journey $(0, x_{j-1})$ in order to satisfy the active constraint $t_{2,j-1} \geq s_j$. Thus our intuitive idea will be that a strategy of optimal type for the following train may use different hold speeds for different sections and as the journey progresses the hold speed will increase.

In this paper we find precise formulæ that allow us to construct these intuitively optimal strategies for the leading and following trains. Note that our imposed constraints ensure that the following train will not enter any particular section until the leading train has left that section.

Keywords: *Optimal train control, train separation, constrained optimization.*

1 THE OPTIMAL DRIVING STRATEGY ON LEVEL TRACK

The equations of motion for a train on a level track $[0, X]$ are

$$vv' = p/v - q - r(v) \tag{1.1}$$

where $x \in [0, X]$ is the position, $v = v(x)$ is the speed, $p = p(x) \in [0, P]$ is the controlled driving power per unit mass, $q = q(x) \in [0, Q]$ is the controlled braking force per unit mass and $r(v)$ is the resistance force per unit mass. The elapsed journey time $t = t(x)$ satisfies the differential equations

$$t' = 1/v \tag{1.2}$$

where $v = v(x)$ is the solution to (1.1). The energy consumption cost is

$$J = \int_0^X (p/v)\, dx \tag{1.3}$$

which is the energy required to drive the train. We have the optimal train control problem on level track.

Problem 1. *Let $T > 0$ be the maximum allowed journey time. Find controls $(p, q) = (p(x), q(x))$ and an associated speed profile $v = v(x)$ satisfying the equation of motion (1.1) with $v(0) = 0$ and $v(X) = 0$ and the elapsed time equation (1.2) and elapsed time constraint $t(X) \le T$ so that the cost (1.3) is minimized.*

If T is sufficiently large then it has been shown (Howlett 2000, Khmelnitsky 2000, Liu and Golovitcher 2003, Albrecht et al. 2013) that the optimal strategy exists and is unique. The solution takes a characteristic form that we call a *strategy of optimal type*. It is convenient to define auxilliary functions $\varphi(v) = vr(v)$ and $\psi(v) = v^2 r'(v)$. Each strategy of optimal type is defined by a designated hold speed $V > 0$ and a corresponding speed $U = \psi(V)/\varphi'(V) = V - \varphi(V)/\varphi'(V) < V$ at which braking begins. There is an associated critical time

$$T_V = \int_0^V \frac{dv}{P/v - r(v)} + \int_U^V \frac{dv}{r(v)} + \int_0^U \frac{dv}{Q + r(v)}. \tag{1.4}$$

If $T > T_V$ then the optimal strategy on level track is a strategy of optimal type (Howlett and Pudney 1995, Howlett 2000) with a phase of *maximum power* driven by controls $(p, q) = (P, 0)$, a phase of *hold* at speed V driven by controls $(p, q) = (\varphi(V), 0)$, a phase of *coast* with controls $(p, q) = (0, 0)$ which terminates when $v(x) = U$ and a phase of *maximum brake* using controls $(p, q) = (0, Q)$. The formula (1.4) defines the collective time taken for the power, coast and brake segments. The corresponding critical distance X_V is given by

$$X_V = \int_0^V \frac{vdv}{P/v - r(v)} + \int_U^V \frac{vdv}{r(v)} + \int_0^U \frac{vdv}{Q + r(v)}. \tag{1.5}$$

Note that both T_V and X_V increase as V increases. The length of the hold phase is chosen to be $X - X_V$ so that the total distance travelled is X and the hold speed V is chosen so that the total elapsed time is T. The time spent in the hold phase is $T - T_V = (X - X_V)/V$.

2 THE EQUATIONS OF MOTION FOR THE TWO TRAIN SEPARATION PROBLEM

The equations of motion for two trains $i = 1, 2$ using different strategies and travelling at different times on the same level track $[0, X]$ are

$$v_i v_i' = p_i/v_i - q_i - r_i(v_i) \tag{2.6}$$

where $x \in [0, X]$ is the position, $v_i = v_i(x)$ is the speed, $p_i = p_i(x) \in [0, P_i]$ is the controlled power per unit mass, $q_i = q_i(x) \in [0, Q_i]$ is the controlled braking force per unit mass, $r_i(v_i)$ is the resistance force per unit mass. The elapsed times $t_i = t_i(x)$ satisfy the differential equations

$$t_i' = 1/v_i \tag{2.7}$$

for each $i = 1, 2$ where $v_i = v_i(x)$ is the solution to (2.6). The energy consumption costs are

$$J_i = \int_0^X (p_i/v_i)\, dx \tag{2.8}$$

for each $i = 1, 2$ which is the energy required to drive the trains. Once again it is convenient to define the auxilliary functions $\varphi_i(v_i) = v_i r_i(v_i)$ and $\psi_i(v_i) = v_i^2 r_i'(v_i)$ for each $i = 1, 2$.

3 THE LEADING TRAIN PROBLEM

For given times $0 = s_0 < \cdots < s_n = T_1$ we wish to find a strategy of optimal type with speed profile $v_1 = v_1(x)$ for a leading train travelling from $x_0 = 0$ to $x_n = X$ with $v_{1,0} = v_1(x_0) = 0$ and $v_{1,n} = v_1(x_n) = 0$ and satisfying the elapsed time constraints $t_{1,0} = s_0$, $t_{1,n} = s_n$ and $t_{1,j} \le s_j$ for each $j = 1, \ldots, n-1$. The leading train satisfies the equations (2.6) and (2.7). If $t_{1,j} = t_1(x_j)$ then

$$t_{1,j} - t_{1,0} = \int_0^{x_j} (1/v_1)\, dx \tag{3.9}$$

for each $j = 1, \ldots, n$. The elapsed time constraints are

$$t_{1,j} - t_{1,0} \le s_j - s_0 \ (j = 1, \ldots, n-1) \quad \text{and} \quad t_{1,n} - t_{1,0} = s_n - s_0. \tag{3.10}$$

Problem 2. *Find controls $(p_1, q_1) = (p_1(x), q_1(x))$ and an associated speed profile $v_1 = v_1(x)$ satisfying the equation of motion (2.6) with $v_1(0) = 0$ and $v_1(X) = 0$ and the elapsed time equation (2.7) and elapsed time constraints (3.10) so that the cost (2.8) is minimized.*

Algorithm LT, below, will find a feasible strategy for the leading train if such a strategy exists.

Algorithm LT is used to construct a journey starting from position x_k at time $t_{1,k} = s_k$ and speed W_k:

Calculate a speed profile $v_1 = v_1(x)$ on $[x_k, x_n]$ satisfying both equation (2.6) with $v_{1,k} = v_1(x_k) = W_k$ and $v_{1,n} = 0$ and equation (2.7) and the time constraint $t_{1,n} - t_{1,k} = s_n - s_k$ in such a way that $J_{1,k} = \int_{x_k}^{x_n} (p_1/v_1)\, dx$ is minimized.

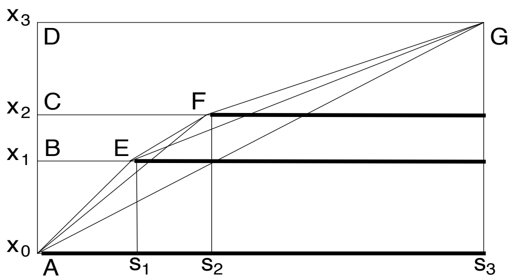
if $t_{1,j} > s_j$ for some $j \in \{k+1, \ldots, n-1\}$ **then**

Find the largest $\ell \in \{k+1, \ldots, n-1\}$ for which the optimal speed profile $\widehat{v}_1$ on $[x_k, x_n]$ satisfying the additional time constraint $\widehat{t}_{1,\ell} - t_{1,k} = s_\ell - s_k$ has no timing point $j \in \{k+1, \ldots, \ell-1\}$ for which $\widehat{t}_{1,j} - t_{1,k} > s_j - s_k$.

Use **Algorithm LT** to calculate a journey starting from position x_ℓ at time $t_{1,\ell}$ and speed $W_\ell = \widehat{v}_1(x_\ell)$.

end if

Algorithm LT is illustrated with a specific example in Fig. 1 where the speed profiles are represented as straight lines with slope equal to the *average speed*. The profiles should not be interpreted literally. Optimal profiles are uniquely determined by the *hold speed* but in general will not be straight lines. In this example LT finds successive optimal speed profiles $\{AG\}$ on $[x_0, x_3]$; $\{AF, FB\}$ on $[x_0, x_3]$ through time-window CF at x_2; and $\{AE, EG\}$ on $[x_0, x_3]$ through BE at x_1 but finds violated time constraints on each occasion. LT then accepts profile $\{AE\}$ on $[x_0, x_1]$ and seeks a profile $\{EG\}$ on $[x_1, x_3]$ (in an overall profile $\{\{AE\}, \{EG\}\}$ through BE at x_1) but finds a violated constraint at x_2; and so (finally) finds profile $\{EF, FG\}$ on $[x_1, x_3]$ though CF at x_2 (in an overall profile $\{\{AE\}, \{EF, FG\}\}$. This example suggests that the optimal strategy for a leading train will use a decreasing sequence of hold speeds on successive intervals.



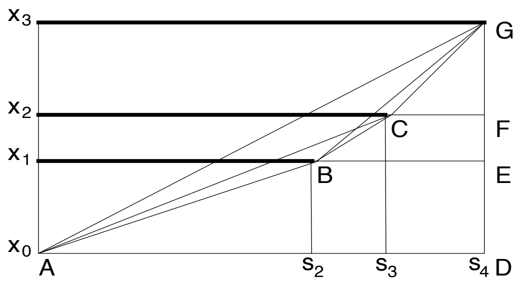


Figure 1: Algorithm LT (left) and Algorithm FT (right).

If Algorithm LT finds a feasible strategy then we can show that an optimal strategy exists. Define arbitrarily chosen non-negative numbers $\{\epsilon_j\}_{j=1}^{n-1}$ such that $s_{j-1} < s_j - \epsilon_j$ for each $j = 1, , \ldots, n-1$ and non-negative speeds $\{W_j\}_{j=1}^{n-1}$ such that $W_j \le W$ for all $j = 1, 2, \ldots, n-1$ where W is the maximum allowed speed.

Assume the optimal journeys on $[x_j, x_{j+1}]$ are feasible subject to $v_{1,0} = 0$ and $v_{1,n} = 0$, $t_{1,0} = s_0$ and $t_{1,n} = s_n$, $v_{1,j} = W_j$ and $t_{1,j} = s_j - \epsilon_j$ for each $j = 1, \ldots, n-1$. Then there is a unique optimal strategy with controls (p_1, q_1) such that all constraints are satisfied. This strategy is simply the sequential compilation of the optimal strategies on each subinterval $[x_j, x_{j+1}]$ subject to the specified constraints.

Note that the cost of the journey $J_1 = J_1(\boldsymbol{\epsilon}, \boldsymbol{W})$ depends on $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_{n-1})$ and $\boldsymbol{W} = (W_1, \ldots, W_{n-1})$ for all feasible arguments $(\boldsymbol{\epsilon}, \boldsymbol{W})$. Let

$$\mathcal{F}_1 \subset \bigtimes_{j=1}^{n-1} [0, s_j - s_{j-1}] \times [0, W]^{n-1} \subset \mathbb{R}^{2n-2} \tag{3.11}$$

be the set of all feasible arguments. Assume that $\mathcal{F}_1 \neq \phi$. The continuous dependence of $v_1 = v_1(x)$ and $t_1 = t_1(x)$ on the boundary conditions defined by $(\boldsymbol{\epsilon}, \boldsymbol{W})$ means that $\mathcal{F}_1$ is closed and hence compact. Since $J_1 = J_1(\boldsymbol{\epsilon}, \boldsymbol{W})$ is also continuous there is a uniquely defined minimum value $J_1^\star$ for J_1 and there is some point $(\boldsymbol{\epsilon}^\star, \boldsymbol{W}^\star)$ with $J_1^\star = J_1(\boldsymbol{\epsilon}^\star, \boldsymbol{W}^\star)$ although this argument does not guarantee uniqueness.

4 The leading train problem on level track

If we make an intuitive assumption about the nature of the optimal strategy—as a succession of optimal strategies in the form *power-hold-coast* on (x_0, x_1) followed by *coast-hold-coast* on each segment (x_j, x_{j+1}) with a decreasing sequence of hold speeds $\{V_j\}$ for $j = 1, \ldots, n-2$ and *coast-hold-coast-brake* on (x_{n-1}, x_n)—then we can formulate the leading train problem on level track in the following way.

Problem 3. *We wish to find an optimal driving strategy for a leading train to travel from $(v, t, x) = (0, 0, 0)$ to $(v, t, x) = (0, T, X)$ on level track subject to the additional time constraints that the train must pass the intermediate points $\{x_j\}_{j=1}^{n-1}$ prior to the given corresponding intermediate times $\{s_j\}_{j=1}^{n-1}$. Suppose that all intermediate constraints are active. That is for $0 \le k < \ell \le n-1$ suppose that the optimal journey from $x = x_k$ to $x = x_{\ell+1}$ subject to the single time constraint $t_{1,\ell+1} - t_{1,k} \le s_{\ell+1} - s_k$ is not feasible if the intermediate constraints $t_{1,j} \le s_j$ for $j = k+1, \ldots, \ell$ are imposed. Our intuitive analysis suggests that we should expect different hold speeds V_j on each interval (x_j, x_{j+1}) with $V_j > V_{j+1}$ for each $j = 0, \ldots, n-1$. We propose a strategy with (1) a power phase terminating at speed V_0 followed by a hold phase at speed V_0 until $x = \alpha_0 \in (x_0, x_1)$ and a coast phase to speed U_1 at $x = x_1$ on (x_0, x_1), (2) a coast phase from speed U_j at $x = x_j$ to speed V_j followed by a hold phase at speed V_j until $x = \alpha_j \in (x_j, x_{j+1})$ and a coast phase to speed U_{j+1} at $x = x_{j+1}$ on (x_j, x_{j+1}) for each $j = 1, \ldots, n-2$ and (3) a coast phase from speed U_{n-1} at $x = x_{n-1}$ to speed V_{n-1}, a hold phase at speed V_{n-1} terminating at $x = \alpha_{n-1} \in (x_{n-1}, x_n)$, a coast phase to speed U_n and a brake phase to speed 0 on (x_{n-1}, x_n). The cost of the strategy is $J = J(\boldsymbol{V}, \boldsymbol{\alpha}, \boldsymbol{U})$ where*

$$\begin{aligned} J \;=\; & \int_0^{V_0} \frac{P dv}{P/v - r(v)} + r(V_0) \left[\alpha_0 - \int_0^{V_0} \frac{v dv}{P/v - r(v)} \right] \\ & \qquad + \sum_{j=1}^{n-1} r(V_j) \left[\alpha_j - x_j - \int_{V_j}^{U_j} \frac{v dv}{r(v)} \right]. \end{aligned} \tag{4.12}$$

The distance travelled between the constraints is given by

$$\delta_j = \alpha_j - x_j + \int_{U_{j+1}}^{V_j} \frac{v dv}{r(v)} \tag{4.13}$$

for each $j = 0, \ldots, n-2$ and

$$\delta_{n-1} = \alpha_{n-1} - x_{n-1} + \int_{U_n}^{V_{n-1}} \frac{v dv}{r(v)} + \int_0^{U_n} \frac{v dv}{Q + r(v)}. \tag{4.14}$$

The corresponding elapsed times are

$$\tau_0 = \int_0^{V_0} \frac{dv}{P/v - r(v)} + \frac{1}{V_0} \left[\alpha_0 - \int_0^{V_0} \frac{v dv}{P/v - r(v)} \right] + \int_{U_1}^{V_0} \frac{dv}{r(v)}, \tag{4.15}$$

$$\tau_j = \int_{U_{j+1}}^{U_j} \frac{dv}{r(v)} + \frac{1}{V_j} \left[\alpha_j - x_j - \int_{V_j}^{U_j} \frac{v dv}{r(v)} \right] \tag{4.16}$$

for each $j = 1, \ldots, n-2$ *and*

$$\tau_{n-1} = \int_{U_n}^{U_{n-1}} \frac{dv}{r(v)} + \frac{1}{V_{n-1}} \left[\alpha_{n-1} - x_{n-1} - \int_{V_{n-1}}^{U_{n-1}} \frac{vdv}{r(v)} \right] + \int_0^{U_n} \frac{dv}{Q + r(v)}. \tag{4.17}$$

We wish to minimize J *subject to the constraints* $\delta_j \geq x_{j+1} - x_j$ *and* $\tau_j \leq s_{j+1} - s_j$ *for each* $j = 0, \ldots, n-1$.

Define a Lagrangian function $\mathcal{J} = J + \sum_{j=0}^{n-1} [\lambda_j(x_{j+1} - x_j - \delta_j) + \mu_j(\tau_j - s_{j+1} + s_j)]$ and calculate

$$\frac{\partial \mathcal{J}}{\partial V_j} = 0 \Rightarrow \left(r'(V_j) - \frac{\mu_j}{V_j^2} \right) \left[\alpha_j - x_j - \int_{V_j}^{U_j} \frac{vdv}{r(v)} \right] + \frac{\varphi(V_j) - (\lambda_j V_j - \mu_j)}{r(V_j)} = 0 \tag{4.18}$$

and

$$\frac{\partial \mathcal{J}}{\partial \alpha_j} = 0 \Rightarrow \frac{\varphi(V_j) - (\lambda_j V_j - \mu_j)}{V_j} = 0 \tag{4.19}$$

for each $j = 0, \ldots, n-1$. We also have

$$\frac{\partial \mathcal{J}}{\partial U_j} = 0 \Rightarrow \frac{(\lambda_{j-1} - r(V_j) - \mu_j / V_j) U_j - (\mu_{j-1} - \mu_j)}{r(U_j)} = 0 \tag{4.20}$$

for each $j = 1, \ldots, n-1$. Finally we have

$$\frac{\partial \mathcal{J}}{\partial U_n} = 0 \Rightarrow \frac{Q(\lambda_{n-1} U_n - \mu_{n-1})}{r(U_n)\,[Q + r(U_n)]} = 0. \tag{4.21}$$

From (4.18) and (4.19) it follows that $y = \lambda_j v - \mu_j$ is tangent to the curve $y = \varphi(v)$ at $v = V_j$ and that $\lambda_j = \varphi'(V_j)$ and $\mu_j = \psi(V_j)$ for each $j = 0, \ldots, n-1$. It follows from (4.20) and (4.21) that

$$U_j = \frac{\psi(V_{j-1}) - \psi(V_j)}{\varphi'(V_{j-1}) - \varphi'(V_j)} \; (j = 1, \ldots, n-1) \quad \text{and} \quad U_n = \frac{\psi(V_{n-1})}{\varphi'(V_{n-1})}. \tag{4.22}$$

It is easy to show that the convexity of $\varphi(v)$ means that $V_j \leq U_j \leq V_{j-1}$ and $U_n \leq V_{n-1}$.

Example 1 (Leading train problem)**.** Let $P = 3$ m^{2}s^{-3}, $Q = 0.3$ ms^{-2}, $r(v) = 6.75 \times 10^{-3} + 5 \times 10^{-5} v^2$ ms^{-2}, $\boldsymbol{x} = (0, 20, 84, 132, 144) \times 10^3$ m, $\boldsymbol{s} = (0, 12, 36, 60, 72) \times 10^2$ s. We used MATLAB to calculate $\boldsymbol{V} = (23.5634, 23.5634, 20.1442, 10.9013)$ ms^{-1}, $\boldsymbol{U} = (23.5634, 21.8984, 15.9814, 5.2714)$ ms^{-1}, $\boldsymbol{t} = (0, 8.82, 36.00, 60.00, 72.00) \times 10^2$ s. Note $\boldsymbol{t} \leq \boldsymbol{s}$ as required.

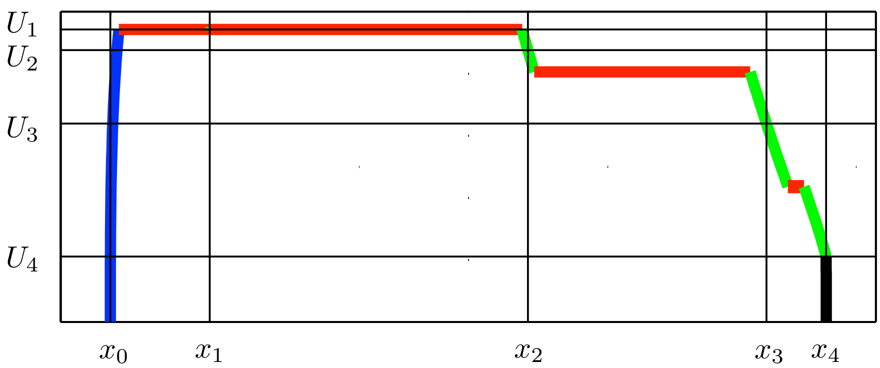


Figure 2: Example 1—optimal strategy *power, hold, coast, hold, coast, hold, coast, brake* for leading train.

5 THE FOLLOWING TRAIN PROBLEM

The analysis follows a similar pattern to the analysis for the leading train problem with some minor differences. For the given times $0 < s_1 \cdots < s_{n+1} = T_2 + s_1$ we wish to find a strategy of optimal type with speed profile $v_2 = v_2(x)$ for a following train travelling from $x_0 = 0$ to $x_n = X$ with $v_{2,0} = v_2(x_0) = 0$ and

$v_{2,n} = v_2(x_n) = 0$ and satisfying the elapsed time constraints $t_{2,0} = s_1$, $t_{2,n} = s_{n+1}$ and $t_{2,j} \geq s_{j+1}$ for each $j = 1, \ldots, n-1$. The following train satisfies the equations (2.6) and (2.7). If $t_{2,j} = t_2(x_j)$ then

$$t_{2,j} - t_{2,0} = \int_0^{x_j} (1/v_2)\, dx \tag{5.23}$$

for each $j = 0, \ldots, n$. The elapsed time constraints are

$$t_{2,j} - t_{2,0} \geq s_{j+1} - s_1 \ (j = 1, \ldots, n-1) \quad \text{and} \quad t_{2,n} - t_{2,0} = s_{n+1} - s_1. \tag{5.24}$$

Problem 4. *Find controls $(p_2, q_2) = (p_2(x), q_2(x))$ and an associated speed profile $v_2 = v_2(x)$ satisfying the equation of motion (2.6) with $v_2(0) = 0$ and $v_2(X) = 0$ and the elapsed time equation (2.7) and elapsed time constraints (5.24) in such a way that the cost (2.8) is minimized.*

Algorithm FT is similar to Algorithm LT and will find a feasible strategy for the following train if such a strategy exists. Algorithm FT is illustrated with a specific example in Fig. 1. In this example FT finds successive optimal speed profiles $\{AG\}$ on $[x_0, x_3]$; $\{AC, CG\}$ on $[x_0, x_3]$ through time-window CF at x_2; and $\{AB, BG\}$ on $[x_0, x_3]$ through BE at x_1 but finds violated time constraints on each occasion. FT then accepts profile $\{AB\}$ on $[x_0, x_1]$ and seeks a profile $\{BG\}$ on $[x_1, x_3]$ but finds a violated constraint at x_2; and so (finally) finds profile $\{BC, CG\}$ on $[x_1, x_3]$ through CF at x_2. This example leads us to expect that the following train will use an increasing sequence of hold speeds on successive intervals.

6 THE FOLLOWING TRAIN PROBLEM ON LEVEL TRACK

If we make an intuitive assumption about the nature of the optimal strategy—as a succession of optimal strategies in the form *power-hold-power* on each segment (x_j, x_{j+1}) with an increasing sequence of hold speeds $\{V_j\}$ for $j = 0, \ldots, n-2$ and *power-hold-coast-brake* on (x_{n-1}, x_n)—we can formulate the leading train problem on level track in the following way.

Problem 5. *We wish to find an optimal driving strategy for a following train to travel from $(v, t, x) = (0, s_1, 0)$ to $(v, t, x) = (0, T + s_1, X)$ on level track subject to the additional time constraints that the train must pass the intermediate points $\{x_j\}_{j=1}^{n-1}$ after some given corresponding intermediate times $\{s_{j+1}\}_{j=1}^{n-1}$. Suppose that all intermediate constraints are active. That is for $0 \leq k < \ell \leq n-1$ suppose that the optimal journey from $x = x_k$ to $x = x_{\ell+1}$ subject to the single time constraint $t_{2,\ell+1} - t_{2,k} \leq s_{\ell+2} - s_{k+1}$ is not feasible if the intermediate constraints $t_{2,j} \geq s_{j+1}$ for $j = k+1, \ldots, \ell$ are imposed. Our intuitive analysis suggests that we should expect different hold speeds V_j on each interval (x_j, x_{j+1}) with $V_j < V_{j+1}$ for each $j = 0, \ldots, n-2$. We propose a strategy with (1) a power phase terminating at speed V_0 followed by a hold phase at speed V_0 until $x = \alpha_0$ and a power phase to speed U_1 at $x = x_1$ on (x_0, x_1), (2) a power phase from speed U_j at $x = x_j$ to speed V_j followed by a hold phase at speed V_j until $x = \alpha_j$ and a power phase to speed U_{j+1} at $x = x_{j+1}$ on (x_j, x_{j+1}) for each $j = 1, \ldots, n-2$ and (3) a power phase from speed U_{n-1} at $x = x_{n-1}$ to speed V_{n-1}, a hold phase at speed V_{n-1} terminating at $x = \alpha_{n-1}$, a coast phase to speed U_n and a final brake phase to speed 0 on (x_{n-1}, x_n). The cost of the strategy is given by $J = J(\boldsymbol{V}, \boldsymbol{\alpha}, \boldsymbol{U})$ where*

$$\begin{aligned} J \;=\; & \int_0^{V_{n-1}} \frac{Pvdv}{P - \varphi(v)} + r(V_0)\left[\alpha_0 - \int_0^{V_0} \frac{v^2 dv}{P - \varphi(v)}\right] \\ & + \sum_{j=1}^{n-1} r(V_j)\left[\alpha_j - x_j - \int_{U_j}^{V_j} \frac{v^2 dv}{P - \varphi(v)}\right]. \end{aligned} \tag{6.25}$$

The distance travelled between the constraints is given by

$$\delta_j = \alpha_j - x_j + \int_{V_j}^{U_{j+1}} \frac{v^2 dv}{P - \varphi(v)} \tag{6.26}$$

for each $j = 0, \ldots, n-2$ and

$$\delta_{n-1} = \alpha_{n-1} - x_{n-1} + \int_{U_n}^{V_{n-1}} \frac{vdv}{r(v)} + \int_0^{U_n} \frac{vdv}{Q + r(v)}. \tag{6.27}$$

The corresponding elapsed times are

$$\tau_0 = \int_0^{U_1} \frac{v dv}{P - \varphi(v)} + \frac{1}{V_0} \left[\alpha_0 - \int_0^{V_0} \frac{v^2 dv}{P - \varphi(v)} \right], \tag{6.28}$$

$$\tau_j = \int_{U_j}^{U_{j+1}} \frac{v dv}{P - \varphi(v)} + \frac{1}{V_j} \left[\alpha_j - x_j - \int_{U_j}^{V_j} \frac{v^2 dv}{P - \varphi(v)} \right] \tag{6.29}$$

for each $j = 1, \ldots, n-2$ *and*

$$\begin{aligned} \tau_{n-1} &= \int_{U_{n-1}}^{V_{n-1}} \frac{v dv}{P - \varphi(v)} + \frac{1}{V_{n-1}} \left[\alpha_{n-1} - x_{n-1} - \int_{U_{n-1}}^{V_{n-1}} \frac{v dv}{P - \varphi(v)} \right] \\ &\qquad + \int_{U_n}^{V_{n-1}} \frac{dv}{r(v)} + \int_0^{U_n} \frac{dv}{Q + r(v)}. \end{aligned} \tag{6.30}$$

We wish to minimize J *subject to the constraints* $\delta_j \geq x_{j+1} - x_j$, $\tau_j \geq s_{j+2} - s_{j+1}$ *for each* $j = 0, \ldots, n-2$, $\delta_{n-1} \geq x_n - x_{n-1}$ *and* $\tau_0 + \cdots + \tau_{n-1} \leq T$.

Define a Lagrangian function

$$\begin{aligned} \mathcal{J} &= J + \sum_{j=0}^{n-2} [\lambda_j (x_{j+1} - x_j - \delta_j) + \mu_j (s_{j+2} - s_{j+1} - \tau_j)] \\ &\qquad + \lambda_{n-1}(x_n - x_{n-1} - \delta_{n-1}) + \mu_{n-1}(\tau_0 + \cdots + \tau_{n-1} - T) \end{aligned}$$

and solve the equations $\partial \mathcal{J}/\partial V_j = 0$ and $\partial \mathcal{J}/\partial \alpha_j = 0$ for each $j = 0, \ldots, n-1$ and the equations $\partial \mathcal{J}/\partial U_j = 0$ for each $j = 1, \ldots, n-1$. Similar formulae to those obtained in the leading train problem now show that $y = \lambda_j v - (\mu_n - \mu_j)$ is tangent to the curve $y = \varphi(v)$ at $v = V_j$ and that $\lambda_j = \varphi'(V_j)$ and $\mu_n - \mu_j = \psi(V_j)$ for each $j = 0, \ldots, n-2$ and that

$$U_j = \frac{\psi(V_j) - \psi(V_{j-1})}{\varphi'(V_j) - \varphi'(V_{j-1})} \quad (j = 1, \ldots, n-1) \quad \text{and} \quad U_n = \frac{\psi(V_{n-1})}{\varphi'(V_{n-1})}. \tag{6.31}$$

It is easy to show that the convexity of $\varphi(v)$ means that $V_{j-1} \leq U_j \leq V_j$ and $U_n \leq V_{n-1}$.

Example 2 (Following train problem)**.** Let $P = 3$ m^{2}s^{-3}, $Q = 0.3$ ms^{-2}, $r(v) = 6.75 \times 10^{-3} + 5 \times 10^{-5} v^2$ ms^{-2}, $\boldsymbol{x} = (0, 20, 84, 132, 144) \times 10^3$ m, $\boldsymbol{s} = (0, 12, 36, 60, 72) \times 10^2$ s. We used MATLAB to calculate $\boldsymbol{V} = (8.2085, 26.4822, 26.4822, 26.4822)$ ms^{-1}, $\boldsymbol{U} = (18.9496, 26.4822, 26.4822, 16.5902)$ ms^{-1}, $\boldsymbol{t} = (12.00, 36.00, 60.27, 78.39, 84.00) \times 10^2$ s. Note $\boldsymbol{t} \geq \boldsymbol{s} + s_1 \times \boldsymbol{1}$ as required.

7 Conclusions and future work

For a given set of intermediate time constraints we found driving strategies for both the leading and following trains on level track which minimize fuel consumption for each train and ensure that the leading train leaves each particular section by the appointed time and the following train enters after the appointed time. There are two major research tasks remaining. What is the best way to find the optimal set of intermediate times that minimize total fuel consumption? What are the optimal strategies on tracks that contain steep sections?

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