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Train scheduling and cooperative games

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Abstract: We wish to determine how train operators should be rewarded when the sequence of trains using a rail corridor is changed from an existing plan to a more efficient plan. The time that each train finishes its journey depends on its departure time and on the progress of the trains ahead of it. Rearranging the order in which the trains depart can increase the lateness cost of some trains and reduce the lateness cost of others. However, the train sequence that minimises the total cost of lateness across all trains may not benefit each individual train. We need to determine a fair redistribution of the overall benefit.

The problem can be formulated as a cooperative game. We consider all possible ways that train operators may choose to rearrange the sequence of trains, from which we can calculate the value of every possible coalition of train operators and hence the set of fair payoffs for changing to a new optimal sequence.

Keywords: *Train scheduling, cooperative game theory.*

1 INTRODUCTION

Rail corridors shared by several competing train operators often have train movements controlled by an independent train controller. As demands for train services change, opportunities arise to improve the performance of the rail network by rescheduling trains.

We consider the case of trains travelling in the same direction along a rail corridor where overtaking is undesirable or not possible. The time at which each train emerges from the end of the corridor can depend on the sequence of trains. In particular, fast trains can be delayed by slower trains ahead of them, or trains can be delayed by preceding trains that depart late. Each train has a deadline for finishing its trip and a cost of being late; the lateness cost is zero for trains that arrive early or on time. We can sometimes get a better overall solution by rearranging trains, but some individual trains may be disadvantaged. How should the overall benefit of an improved train sequence be distributed amongst the train operators?

We will illustrate the key features of the problem using an example with four trains travelling in one direction along a corridor where overtaking is not permitted. Each train is characterised by the time r_i that it is ready to start its trip and the minimum possible trip duration d_i. The target completion time for each trip is $t_i = r_i + d_i$, and the cost of lateness is $c_i = \alpha_i \max\{0, f_i - t_i\}$ where α_i is the lateness cost rate for the trip and f_i is the actual finish time. This is a highly simplified version of the train scheduling problem but the principles are applicable to more realistic scheduling formulations.

Suppose the original sequence of the four trains on the corridor is ABCD and the timetabling requirements have changed to:

$$(r_\mathrm{A}, d_\mathrm{A}) = (2, 5) \quad (r_\mathrm{B}, d_\mathrm{B}) = (0, 5) \quad (r_\mathrm{C}, d_\mathrm{C}) = (0, 7) \quad (r_\mathrm{D}, d_\mathrm{D}) = (0, 6).$$

This example has a fast train following a slow train and a train delayed at the start.

A train cannot start before its ready time. A minimum headway of 1 is required between trains at the start of the corridor and at the end of the corridor. Table 1 shows the calculation of start time s_i, finish time f_i and lateness cost c_i for each train, and the total lateness cost C for the train sequence ABCD. The highlighted numbers in column s_i indicate starting delays due to the late departure of a preceding train; the highlighted number in column f_i indicates a finishing delay due to a preceding slow train.

train	r_i	s_i	d_i	t_i	f_i	c_i
A	2	2	5	7	7	0
B	0	**3**	5	5	8	3
C	0	**4**	7	7	11	4
D	0	**5**	6	6	**12**	6
						13

Table 1. Calculation of train costs for our four-train example with train sequence ABCD.

In general, changing the sequence of trains will change the lateness cost of each train. Some costs will increase, others will decrease. Table 2 shows the costs of each train for train sequences where the total cost is not more than the total cost of the train sequence ABCD. There are three sequences where the total cost increased; these are not shown. The columns headed g_i indicate the gain for each train; that is, the reduction in cost from the cost in the initial sequence ABCD. The final column is the total gain, G.

How should gains achieved by changing the train sequence be distributed amongst the train operators?

2 COOPERATIVE GAME THEORY

We will use concepts from cooperative game theory to determine how the total gain of a rearranged train sequence should be distributed. We will refer to the owners of the trains as *players*. One way to redistribute the total gain is to ensure that each player has the same *payoff*, $x_i = G/n$; the payoff for a player is the reduction in the player's cost after gains have been redistributed.

train sequence	c_{A}	c_{B}	c_{C}	c_{D}	C	g_{A}	g_{B}	g_{C}	g_{D}	G
BDAC	1	0	3	1	5	-1	3	1	5	8
DBAC	1	2	3	0	6	-1	1	1	6	7
BDCA	3	0	2	1	6	-3	3	2	5	7
BADC	0	0	4	3	7	0	3	0	3	6
DABC	0	3	4	0	7	0	0	0	6	6
BCAD	2	0	1	4	7	-2	3	3	2	6
BCDA	3	0	1	3	7	-3	3	3	3	6
DBCA	3	2	2	0	7	-3	1	2	6	6
BACD	0	0	3	5	8	0	3	1	1	5
DCAB	2	5	1	0	8	-2	-2	3	6	5
DCBA	3	4	1	0	8	-3	-1	3	6	4
CABD	1	4	0	4	9	-1	-1	4	2	4
CADB	1	5	0	3	9	-1	-2	4	3	4
DACB	0	6	3	0	9	0	-3	1	6	4
CBAD	2	3	0	4	9	-2	0	4	2	4
CDAB	2	5	0	2	9	-2	-2	4	4	4
CBDA	3	3	0	3	9	-3	0	4	3	4
CDBA	3	4	0	2	9	-3	-1	4	4	4
ABDC	0	3	5	4	12	0	0	-1	2	1
ABCD	0	3	4	6	13	0	0	0	0	0
ADBC	0	5	5	3	13	0	-2	-1	3	0

Table 2. Cost and gains of each feasible train sequence for our example problem.

There are some basic properties that are considered desirable in cooperative games:

- **individual rationality**: no player gets less than they could get by acting alone
- **efficiency**: the total gain is distributed amongst the players
- **symmetry**: equivalent players get the same payoff.

Distributing the total gain equally amongst all players does not necessarily meet all of these requirements. For example, consider a scenario with two trains, A and B with equal speeds, where A has a ready time greater than the headway. Train B can go before A without affecting A, and so should be able to keep the entire gain.

In cooperative game theory, additional fairness properties are often defined which further constrain the set of possible payoffs. To define such properties, we first need a precise definition of our game. Games are usually defined by a *characteristic function* which gives a value to every possible coalition of players, or by a *partition function* which gives a value to every coalition in every possible partition of players into coalitions. The payoff to each player is then determined by a process that considers the marginal contribution of each player to every coalition (e.g. Shapley (1971)) or considers all possible ways that partitions could form (e.g. Maskin (2003)). In our case we know the value of each train sequence but we must define a method for determining the value of each possible coalition.

We define a value function v that specifies the maximum value $v(S)$ that can be achieved by a coalition S of players by admissible rearrangements (Curiel et al., 1989, 1994; Hamers et al., 1995; Borm et al., 2002; Calleja et al., 2006). From Table 2 it is clear that the coalition of players $\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ can rearrange itself into the sequence BDAC and generate a value $v(\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}) = 8$. For conciseness, we will write this as $v(\mathrm{ABCD}) = 8$. To determine the value of an arbitrary coalition, we need to know the allowable actions of a coalition, and the value to the coalition of each possible action.

At any time, any player i in any coalition S may remove itself from the train sequence then insert itself back into the sequence at any position, provided that:

- the total gain of the players in S is non-negative

- no player outside S has a negative gain.

We also need to consider the situation where a player outside a coalition S has a positive gain (called a 'windfall' gain) from the movement of a player inside S. There are two possible variants of admissible movements:

- **with windfall gains**: players outside a coalition S may benefit from the movement of a player inside S
- **without windfall gains**: players outside a coalition S may not benefit from the movement of a player inside S, and so moves that would result in windfall gains are not permitted.

In our example, if B moves to change the train sequence from ABCD to BACD then B gains 3, but C and D also each gain 1. If windfall gains are not permitted then players B, C and D must all be in the same coalition for the move to be permitted, and may redistribute the total gain of 5 amongst themselves. If windfall gains are permitted then B may move even if C or D are not in the same coalition as B, but gains to players outside S cannot be distributed amongst the players within S.

Using these rules, we can now define the value of each coalition. This in turn enables us to define the *core* $C(v)$—a set of payoff vectors for which the payoff for each player is at least as great as the payoff they can receive as a member of any coalition in game v, that is:

$$C(v) = \left\{ \mathbf{x} \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq \ v(S) \quad \forall S \subseteq N \right\} \tag{1}$$

where N is the grand coalition of all players.

We illustrate the calculation of coalition values before describing the general method. First, consider the case where windfall gains are not permitted. The possible moves between sequences are shown in Figure 1. Each arc is labelled by the name of the moving player, and the gains to each of the players before any redistribution.

Figure 1. Graph of admissible moves for the example problem when windfall gains are not permitted. Details are shown in Table 3.

We can construct a simplified version of this graph for each S. First, we remove inadmissible arcs where:

- the total gain of S is negative
- the moving player is in S and some player outside S has non-zero gain

- the moving player is not in S and some player in S has non-zero gain.

The value of a coalition is the maximum value that the coalition is guaranteed to achieve, no matter what the players outside the coalition do (or don't do).

To help us calculate the value of a coalition we remove all arcs from any state where there are no moves from coalition players from that state, since the coalition cannot force a move from such states. We also remove any states that cannot be reached from the initial sequence. The value of the coalition S can be calculated from the remaining graph. The non-empty graphs are shown in Table 3.

If no graph remains then the value of the coalition is zero. For each of the remaining graphs in our example without windfall gains, all of the arcs belong to the coalition and so the coalition can achieve any of the values of the sequences in the graph; the value of the coalition is the maximum of these sequence values.

When windfall gains are permitted, the inadmissible arcs are those arcs where:

- the total gain of S is negative
- the moving player is in S and some player outside S has negative gain
- the moving player is not in S and some player in S has negative gain.

Tables 4–6 show the graphs and values for all coalitions when windfall gains are permitted. In this case, graphs contain arcs by players outside the coalition and so calculating the value of the coalition is more complicated.

We will describe some example value calculations before describing a general method for determining the value of a coalition:

- Coalition A has value $v(\mathrm{A}) = 0$ since none of the arcs in the graph (Table 4) have any gain for A.
- Coalition B has value $v(\mathrm{B}) = 0$ since player D could move first to form sequence DABC; B has no gain from this move, and no further moves are possible.
- Coalition D has value $v(\mathrm{D}) = 1$ since that is the minimum gain that D will get, if A or B move to form sequence BACD.
- Coalition AB has value $v(\mathrm{AB}) = 0$ since D could move first to form sequence DABC with no gain for AB and from which AB can not make any gain, or C could move to form ABDC then D could move to form sequence DABC, again with no gains for AB.
- Coalition AD has value $v(\mathrm{AD}) = 4$. If AD were to move first then D would move to form sequence DABC with gain 6, but if B moves first to form sequence BACD (with gain 1 for AD) then D can move for form sequence BDAC with an additional gain of 3 for AC.

In general, the value of a coalition S is the total gains to the coalition from the best sequence that the coalition can force. The total gain made by a coalition S if it can force a move to a sequence σ is

$$G_S(\sigma) = \sum_{i \in S} g_i(\sigma)$$

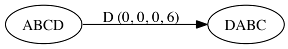
where g_i is the gain to player i (from Table 2, for example).

We say that a sequence σ is a *final sequence for the coalition* S if there are no possible moves in the simplified graph, by any player, from sequence σ to any adjacent sequence σ^* with $G_S(\sigma^*) > G_S(\sigma)$. For example, in Table 5 for coalition AD, the final sequences are DABC, BCAD, BCDA and BDAC; BACD is not a final sequence because D can move to form sequence BDAC, and $G_{\mathrm{AD}}(\mathrm{BDAC}) = 4 > G_{\mathrm{AD}}(\mathrm{BACD}) = 1$.

We can label each sequence σ in a coalition's graph by the best total gain that the coalition can force from that sequence, which we denote $\hat{G}_S(\sigma)$. If σ is a final state then $\hat{G}_S(\sigma) = G_S(\sigma)$, otherwise

$$\hat{G}_S(\sigma) = \min\left\{ \max_{(p,\sigma^*)\in\mu(\sigma), p\in S} \left\{\hat{G}_S(\sigma^*)\right\}, \min_{(p,\sigma^*)\in\mu(\sigma), p\notin S} \left\{\hat{G}_S(\sigma^*)\right\} \right\}$$

where $\mu(\sigma)$ is the set of admissible moves $\{(p, \sigma^*)\}$ by any player p from sequence σ to sequence σ^*, corresponding to arcs of the coalition's graph. The first term in the outer minimisation occurs because the members

<table>
<tr><th>S</th><th>graph</th><th>$v(S)$</th></tr>
<tr><td>D</td><td>
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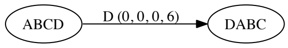
<tr><td>AD</td><td>
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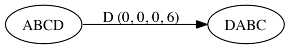
<tr><td>BD</td><td>
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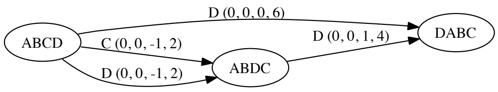
<tr><td>CD</td><td>
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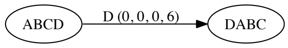
<tr><td>ABD</td><td>
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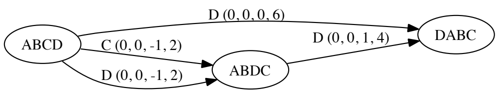
<tr><td>ACD</td><td>
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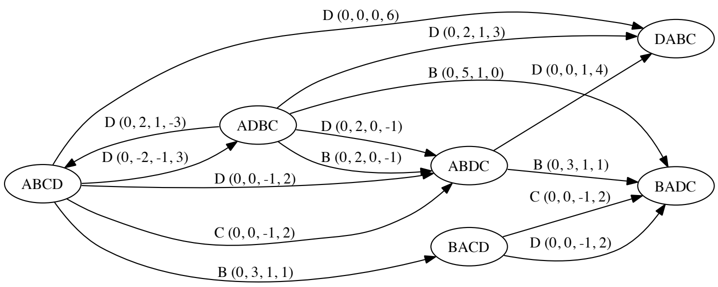
<tr><td>BCD</td><td>
</td><td>6</td></tr>
<tr><td>ABCD</td><td>(see Figure 1)</td><td>8</td></tr>
</table>

Table 3. Coalition values for the four-train example, with windfall gains not permitted. Coalitions not shown have empty graphs with zero value.

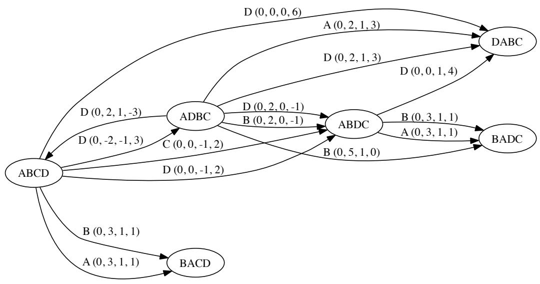
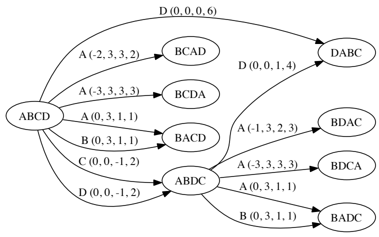
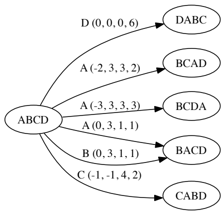
S	graph	$v(S)$
A		0
B		0
C	(empty graph)	0
D		1

Table 4. Coalition values for the four-train example, with windfall gains permitted.

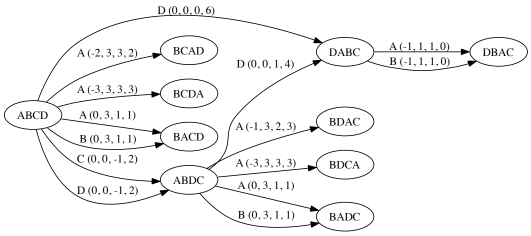
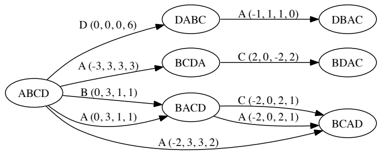
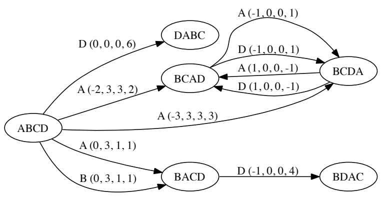
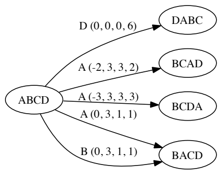
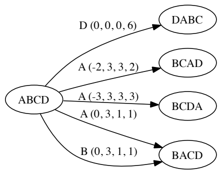
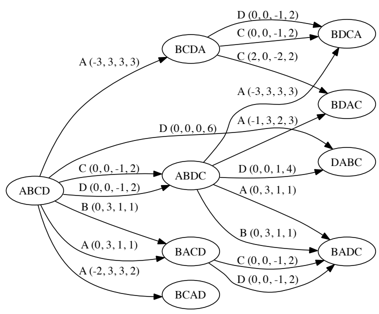
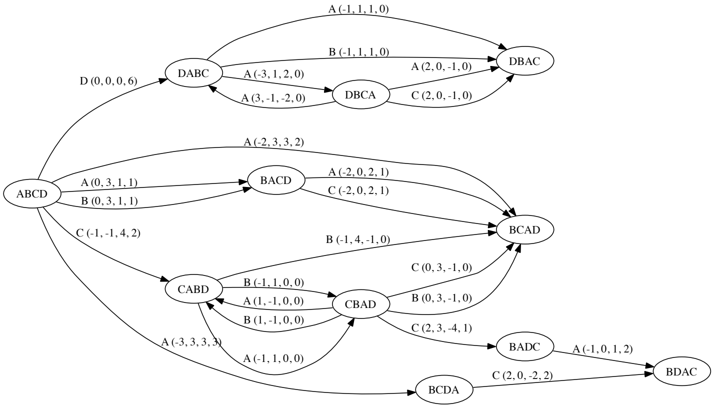
S	graph	$v(S)$
AB		0
AC		0
AD		4
BC		0
BD		4
CD		3

Table 5. More coalition values for the four-train example, with windfall gains permitted.

<table>
<tr><th>S</th><th>graph</th><th>$v(S)$</th></tr>
<tr><td>ABC</td><td>
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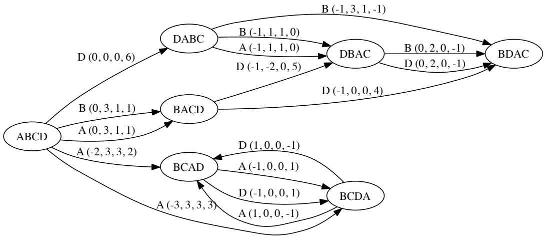
<tr><td>ABD</td><td>
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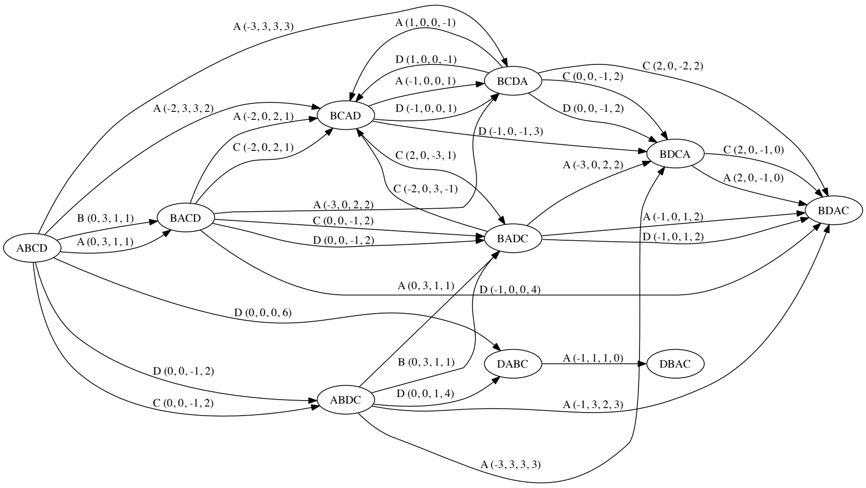
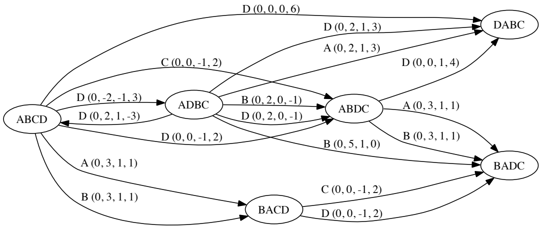
<tr><td>ACD</td><td>
</td><td>5</td></tr>
<tr><td>BCD</td><td>
</td><td>6</td></tr>
<tr><td>ABCD</td><td>(not shown)</td><td>8</td></tr>
</table>

Table 6. More coalition values for the four-train example, with windfall gains permitted.

of the coalition will attempt to maximise their gain $\hat{G}$; the second term occurs because players outside the coalition may frustrate the coalition. We start by labelling the final sequences, then work backwards towards the initial sequence. The value of the coalition S is $v(S) = \hat{G}_S(\sigma_0)$, where σ_0 is the initial sequence.

Figure 2 shows the labelled graph for coalition ABC with windfall gains. The final sequences are DBAC, BCAD, BACD, BDAC, BADC and BCDA. There are looping arcs, with no change of value for the coalition, between sequences DABC and DBCA, and between sequences CABD and CABD. When looping arcs occur, both sequences in the loop will have the same value of G_S but may have different values of $\hat{G}_S$ depending on whether the looping arcs are from players inside or outside the coalition. When a loop occurs between two nodes σ_1 and σ_2, we first calculate $\hat{G}_S(\sigma_1)$ and $\hat{G}_S(\sigma_2)$ without the looping arcs, then replace the looping arcs and update each of $\hat{G}_S(\sigma_1)$ and $\hat{G}_S(\sigma_2)$.

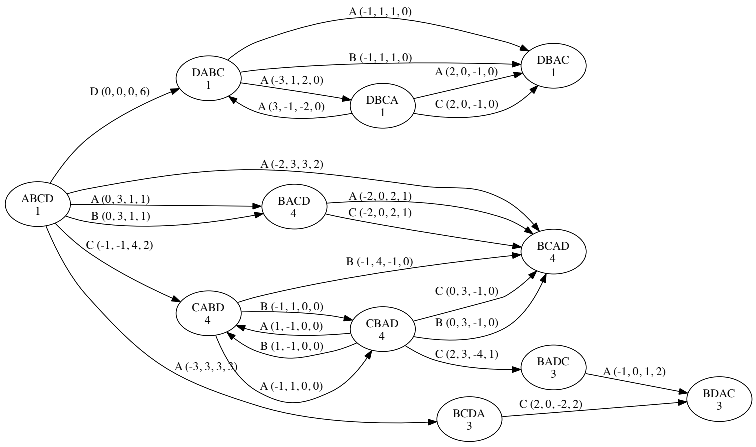


Figure 2. Labelled graph for coalition ABC with windfall gains.

Table 7 summarises the values $v(S)$ for each coalition S, without windfall gains and with windfall gains permitted. We can use these values to calculate the core (equation (1)) of each game and also the Shapley value (Shapley, 1971) of each game:

- without windfall gains, the Shapley value is $(0.5, 0.5, 0.5, 6.5)$
- with windfall gains, the Shapley value is $(1.25, 1.58, 0.75, 4.42)$.

3 SUMMARY

Changing the sequence of trains on a corridor can give an overall reduction in the cost of lateness, but the cost of lateness may increase for individual trains. We have shown how we can use cooperative game theory to calculate payoffs to each train operator so that no operator is unfairly compensated. In particular, we have shown how coalition values can be calculated so that the set of fair payoffs—the core—can be calculated.

S	without windfall gains	with windfall gains
A	0	0
B	0	0
C	0	0
D	6	1
AB	0	0
AC	0	0
AD	6	4
BC	0	0
BD	6	4
CD	6	3
ABC	0	1
ABD	6	7
ACD	6	5
BCD	6	6
ABCD	8	8

Table 7. Coalition values for games without and with windfall gains.

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