An Inventory Model With Shortage and Quantity Dependent Permissible Delay In Payment

Manisha Pal* and Sanjoy Kumar Ghosh

Department of Statistics, University of Calcutta, 35 Ballygunge Circular Road,
Kolkata – 700019, India.

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Abstract

In many inventory situations, instead of making immediate payment on receiving the consignment, the purchaser is allowed a certain fixed time period to pay back the cost of goods bought. This paper studies an inventory model where the permissible delay in payment depends on the ordered quantity. Numerical examples have been cited to illustrate the model.

Keywords and Phrases: Inventory; Constant demand rate; Shortage; Backlogging; Permissible delay in payments.

1. Introduction

In today's business transactions, it is frequently observed that a customer is allowed some grace period before settling the account with the supplier or the producer. The customer does not have to pay any interest during this fixed period but if the payment gets delayed beyond the period interest will be charged by the supplier. This arrangement comes out to be very advantageous to the customer as he may delay the payment till the end of the permissible delay period. During the period he may sell the goods, accumulate revenues on the sales and earn interest on that revenue. Thus, it makes economic sense for the customer to delay the payment of the replenishment account up to the last day of the settlement period allowed by the supplier or the producer. Goyal [2] first developed an economic order quantity (EOQ) model under the condition of permissible delay in payments. Shinn *et al.* [6] extended

* Corresponding author. Tel: 913324252143; Fax: 913324764419
E-mail address: manishapal2@gmail.com

the model by considering quantity discount for freight cost. Recently Aggarwal and Jaggi [1] and Hwang and Shinn [3] extended Goyal's model to consider deterministic inventory model with constant rate of deterioration. Shah and Shah [5] developed probabilistic inventory model for deteriorating items when delay in payment is permitted.

Shortages are of great importance especially in a model that considers a delay in payment due to the fact that shortages can affect the quantity ordered to benefit from the delay in payment. Jamal *et al.* [4] extended Aggarwal and Jaggi's model to allow for shortages. Now there arises a natural question whether the length of the permissible delay period gets influenced by the volume of the quantity ordered. Intuition leads to the fact that the volume of the ordered quantity should have a direct impact on the length of this period. More precisely we can say that the more we order the longer the delay period is likely to be allotted. The present paper incorporates this fact in an inventory model allowing shortages and obtains the optimal ordering policy. The paper is organized as follows. In section 2 assumptions and notations are presented. In section 3 the mathematical model is formulated and some relevant results are proved, based on which an algorithm for finding the optimal policy is suggested. Lastly, in section 4, numerical examples are cited to illustrate the model.

2. Notations and Assumptions

To develop the proposed model, the following notations and assumptions are used in this paper.

2.1. Notations

$I(t)$ = inventory level at time t.

K = ordering cost of inventory per order,

P = per unit purchase cost,

h = per unit holding cost excluding interest charges,

s = per unit shortage cost,

I_e = interest which can be earned,

I_r = interest charges which invested in inventory, $I_r \geq I_e$.

M = permissible delay in settling the accounts, $0 < M < T$.

T = length of the replenishment cycle.

T_1 = time when inventory level comes down to zero, $0 \leq T_1 < T$.

$C_M(T_1, T)$ = average total inventory cost per unit time when permissible delay period in payment is M.

Let us write

$$C_M(T_1, T) = \begin{cases} C^1{}_M(T_1, T) & \text{for } T_1 \geq M \\ C^2{}_M(T_1, T) & \text{for } T_1 < M \end{cases}$$

2.2 Assumptions

1. The inventory system involves only one item.
2. Replenishment occurs instantaneously on ordering i.e. lead-time is zero.
3. Demand rate $R(t)$ is deterministic and given by
   $$R(t) = \alpha \quad ; \qquad 0 < t < T.$$
4. Shortages are allowed and completely backlogged.

5. The planning period is of infinite length. The planning horizon is divided into sub-intervals of length T units. Orders are placed at time points 0, T, $2T$, $3T$,…. the order quantity at each reorder point being just sufficient to bring the stock height to a certain maximum level S.
6. The length of the permissible delay period M for repaying the supplier is given by

$$M = \begin{cases} M_1 & \text{if } 0 < q < q_0 \\ M_2 & \text{if } q > q_0 \end{cases}$$

   where q is the ordered quantity and q_0 a specified value of q, and $M_2 > M_1$.
7. No payment to the supplier is outstanding at the time of placing an order, i.e. $M < T$.

3. Model Formulation

Since the planning period is of infinite length, we study the model over a reorder interval, say (0, T). Two situations can arise, which are described pictorially in figure 1 and figure 2.

Variations of inventory level $I(t)$ w.r.t. time is given by

$$\frac{dI(t)}{dt} = -\alpha \; ; \qquad 0 < t < T \qquad (1)$$

The solution of (1) is

$$I(t) = \alpha(T_1 - T)\; ; \qquad 0 < t < T \qquad (2)$$

with boundary condition $I(T_1) = 0$.

In the interval (0, T_1),

expected holding cost $HC = h\int_0^{T_1} I(t)dt = \frac{\alpha h T_1^2}{2}$

Over the interval (T_1, T),

expected shortage cost $SC = s\int_{T_1}^{T} \alpha(T-t)dt = \alpha s\frac{(T-T_1)^2}{2}$

CASE 1: $M \le T_1$

In this situation since the length of period with positive stock is larger than the credit period, the buyer can use the sale revenue to earn interest at an annual rate I_e in $(0, T_1)$. The interest earned IE_1, is

$$IE_1 = PI_e\int_0^{T_1} \alpha(T_1 - t)dt = \alpha PI_e\frac{T_1^2}{2}$$

However beyond the credit period, the unsold stock is supposed to be financed with an annual rate I_r and the interest payable IP is given by

$$IP = P\,I_r \int_M^{T_1} \alpha\left(T_1 - t\right)dt = \frac{\alpha P I_r}{2}\left(T_1 - M\right)^2$$

Therefore the total average cost per unit time is

$$C^1{}_M(T_1, T) = \frac{K + HC + SC + IP - IE_1}{T}$$

$$= \frac{1}{T}\left[K + \frac{\alpha h T_1^2}{2} + \alpha s\frac{\left(T - T_1\right)^2}{2} + \frac{\alpha P I_r}{2}\left(T_1 - M\right)^2 - \alpha P I_e \frac{T_1^2}{2}\right]$$

Optimal values of T_1 and T which minimize $C^1{}_M(T_1, T)$ are obtained by solving the equations $\frac{dC_M^1(T_1, T)}{dT_1} = 0$ and $\frac{dC_M^1(T_1, T)}{dT} = 0$ which give

$$(h + s + P I_r - P I_e)\, T_1 - s\,T = P I_r M \qquad (3)$$

$$\frac{1}{T}\left[\alpha\, s\,(T - T_1)\right] - \frac{1}{T^2}\left[K + \frac{\alpha h T_1^2}{2} + \alpha s\frac{\left(T - T_1\right)^2}{2} - \alpha P I_e \frac{T_1^2}{2}\right] = 0 \qquad (4)$$

CASE 2: $M > T_1$

Since $M > T_1$, the buyer pays no interest but earns interest at an annual rate I_e during the period $(0, M)$. Interest earned in this case, denoted by IE_2, is given by

$$IE_2 = \alpha P I_e T_1\left(M - \frac{T_1}{2}\right)$$

Then the total average cost per unit time is

$$C^2{}_M(T_1, T) = \frac{K + HC + DC + SC - IE_2}{T}$$

$$= \frac{1}{T}\left[K + \frac{\alpha h T_1^2}{2} + \alpha s\frac{\left(T - T_1\right)^2}{2} - \alpha P I_e T_1\left(M - \frac{T_1}{2}\right)\right]$$

Optimal values of T_1 and T which minimize $C^2{}_M(T_1, T)$ are obtained by solving the equations

$\frac{dC_M^2(T_1, T)}{dT_1} = 0$ and $\frac{dC_M^2(T_1, T)}{dT} = 0$ which give

$$(h + s + P I_e)\, T_1 - s\,T = P I_e M \qquad (5)$$

$$\frac{1}{T}[\ \alpha s\ (\ T\text{-}\ T_1)] - \frac{1}{T^2}[\ K + \frac{\alpha h T_1^2}{2} + \alpha s\frac{(T-T_1)^2}{2} - \alpha P I_e T_1\left(M - \frac{T_1}{2}\right)\ ] = 0 \qquad (6)$$

We observe the following properties of optimal T_1 and T :

RESULT 1 : Optimal T_1 is an increasing function of T.

PROOF : From (3) we have

$$T_{1opt} = \frac{sT + PI_r M}{h + s + P(I_r - I_e)} \qquad (7)$$

Hence,

$$\frac{dT_{1opt}}{dT} = \frac{s}{h + s + P(I_r - I_e)} = C \text{ ; (say),}$$

which is a constant independent of T , since $I_r > I_e$ and $0 < C < 1$.

From (5) we have

$$T^*_{1opt} = \frac{sT + PI_e M}{h + s + PI_e} \qquad (8)$$

Then,

$$\frac{dT^*_{1opt}}{dT} = \frac{s}{h + s + PI_e} = b \text{ (say),}$$

which is a constant independent of T .

Clearly, $0 < b < 1$.

Hence, optimal T_1 is an increasing function of T.

RESULT 2: Optimal T is an increasing function of M.

PROOF : Substituting (7) in $C^1{}_M(T_1, T)$, we get $\min_{T_1} C^1{}_M(T_1, T) = C^1{}_M(T)$,say. Optimal T is obtained by solving $\frac{\partial}{\partial T} C_M^1(T) = 0$, which gives

$$-\frac{N}{T^2} + \frac{\alpha}{T}[s\ (\ T - T_{1opt})\ (\ 1 - C\ ) + h\ T_{1opt} C + P\ I_r\ (\ T_{1opt} - M\ )\ C - P\ I_e\ T_{1opt} C] = 0,$$

where N is the numerator of $C^1{}_M(T)$.

Hence,

$$\frac{N}{T} = \alpha\,[s\,(T\text{-}T_{1opt})\,(1 - C) + h\,T_{1opt}\,C + P\,I_r\,(T_{1opt} - M)\,C - P\,I_e\,T_{1opt}\,C\,]$$

Differentiating the above expression w.r.t. M we get

$$\frac{\partial\left(\frac{N}{T}\right)}{\partial T}\frac{\partial T}{\partial M} = \alpha\,[\,h\;C^2\,\frac{\partial T}{\partial M} + s\,(1 - C)\,(\frac{\partial T}{\partial M} - C\frac{\partial T}{\partial M}) + P\,I_r\,(C\frac{\partial T}{\partial M} -1) - P\,I_e\,\frac{\partial T}{\partial M}\,C^2\,]$$

or, $[\,hC^2 + s\,(1-C)^2 + P\,I_r\,C^2 - P\,I_e\;C^2\,]\;\frac{\partial T}{\partial M} = P\,I_r\,C,$

as $\frac{\partial\left(\frac{N}{T}\right)}{\partial T} = \frac{\partial}{\partial T}C_M^1(T) = 0.$

$$\therefore \frac{\partial T}{\partial M} = \frac{CPI_r}{hC^2 + s(1-C)^2 + P(I_r - I_e)C^2} > 0;\;\text{ since } I_r > I_e.$$

Again, let $\min\limits_{T_1} C^2{}_M(T_1, T) = C^2{}_M(T)$, say.

Optimal T is obtained by solving $\frac{\partial}{\partial T}C_M^2(T) = 0$, which gives

$$-\frac{N}{T^2} + \frac{\alpha}{T}[\,s\,(T - T^*{}_{1opt})\,(1\text{-}b) + h\,T^*{}_{1opt}\,b - P\,I_e\,(M\,b - b\,T^*{}_{1opt})] = 0,$$

where N is the numerator of $C^2{}_M(T)$.

Hence,

$$\frac{N}{T} = \alpha\,[\,h\,T^*{}_{1opt}\;\;b + s\,(T - T^*{}_{1opt})\,(1\text{-}b) - P\,I_e\,b(M - T^*{}_{1opt})]$$

Differentiating the above expression w.r.t. M we get

$$\frac{\partial\left(\frac{N}{T}\right)}{\partial T}\frac{\partial T}{\partial M} = \alpha\,[\,h\,b^2\frac{\partial T}{\partial M} + s\,(1 - b)\,(\frac{\partial T}{\partial M} - b\frac{\partial T}{\partial M}) - P\,I_e\,b\,(1 - b\frac{\partial T}{\partial M})]$$

or, $\quad [\,h\,b^2 + s\left(1-b\right)^2 + P\,I_e\,b^2\,]\;\frac{\partial T}{\partial M} = b\,P\,I_e,$

since $\quad \frac{\partial\left(\frac{N}{T}\right)}{\partial T} = \frac{\partial C^2{}_M\left(T\right)}{\partial T} = 0.$

Therefore,

$$\frac{\partial T}{\partial M} = \frac{bPI_e}{hb^2 + s(1-b)^2 + PI_e b^2} > 0.$$

Hence the result follows.

RESULT 3 : $C^1_M(T)$ is convex in T.

PROOF: $\frac{\partial}{\partial T} C^1_M(T) = 0$ gives

$$-\frac{N}{T^2} + \frac{1}{T}\frac{dN}{dT} = 0$$

or, $N = T\frac{dN}{dT}$, where N is the numerator of $C^1_M(T)$.

So, for any T satisfying $\frac{\partial}{\partial T} C^1_M(T) = 0$,

$$\frac{\partial^2}{\partial T^2} C^1_M(T) = 2\frac{N}{T^3} - \frac{1}{T^2}\frac{dN}{dT} - \frac{1}{T^2}\frac{dN}{dT} + \frac{1}{T}\frac{d^2N}{dT^2}$$

$$= \frac{1}{T}\frac{d^2N}{dT^2}, \text{ (since } N = T\frac{dN}{dT})$$

$$= \frac{1}{T}\,[\, h\, C^2 + s\,(1-C)^2 + P\,(I_r - I_e)\,C^2\,] > 0, \text{ as } I_r > I_e.$$

This is possible only if $C^1_M(T)$ is convex in T.

Hence, the result.

RESULT 4 : $C^2_M(T)$ is convex in T.

PROOF: The proof is similar to that of result 3.

RESULT 5: $C_M(T_1, T)$ is a decreasing function of M.

PROOF : We observe that for given (T_1, T), $C^1_M(T_1, T)$ and $C^2_M(T_1, T)$ are decreasing functions of M.

Consider any M_1 and M_2 such that $M_1 < M_2$.

(i) Suppose $M_1 < M_2 < T_1$.

Then, $C_{M_1}(T_1,T) = C^1_{M_1}(T_1,T) > C^1_{M_2}(T_1,T) = C_{M_2}(T_1,T)$.

(ii) For $M_1 < T_1 < M_2$,

$$C_{M_1}(T_1,T) = C^1_{M_1}(T_1,T) > C^1_{M_2}(T_1,T) > C^2_{M_2}(T_1,T) = C_{M_2}(T_1,T).$$

(iii) For $T_1 < M_1 < M_2$,

$$C_{M_1}(T_1, T) = C^2_{M_1}(T_1, T) > C^2_{M_2}(T_1, T) = C_{M_2}(T_1, T).$$

Hence the result follows.

Based on the above results, we develop the following algorithm to find the optimal values of T_1 and T.

ALGORITHM

Step1: Find (T_1^*, T^*) minimizing $C_{M_2}(T_1, T)$. If $\alpha T^* \geq q_o$, (T_1^*, T^*) is optimal, Else, go to step 2.

Step2: Find (T_1^{**}, T^{**}) minimizing $C_{M_1}(T_1, T)$ and compute $C_{M_1}(T_1^{**}, T^{**})$ and $C_{M_2}(T_1^0, T^0)$, where $\alpha T^0 = q_o$ and T_1^0 is the optimal value of T_1 for given $T = T^0$. If $C_{M_1}(T_1^{**}, T^{**}) < C_{M_2}(T_1^0, T^0)$, then (T_1^{**}, T^{**}) is optimal, else (T_1^0, T^0) is optimal.

4. Numerical Examples

EXAMPLE 1

Let K= \$5.00 per order, h = \$10.00 per unit, P = \$50.00 per unit, s = \$20.00 per unit, α = 300 units, I_r = 0.15, I_e = 0.10 and

$$M = \begin{cases} 15 \text{ days if } 0 < q < 12000 \\ 30 \text{ days if } q \geq 12000 \end{cases}$$

Step1: Consider $M = 30$ days.

For $T_1 \geq 30$, T_{opt} = 37.5 days, T_{1opt} = 30 days and $C^1_{30}(T_{1opt}, T_{opt})$ = \$110.31.

For $T_1 < 30$, T_{opt} = 30 days, T_{1opt} = 22 days and $C^2_{30}(T_{1opt}, T_{opt})$ = \$87.41.

Hence, optimal T and T_1 minimizing $C_{30}(T_1, T)$ are T^* = 30 days, T_1^* = 22 days, and $C_{30}(T_1^*, T^*)$ = \$87.41.

But $\alpha T^* = 9000 < q_0 = 12000$. So we go to the next step.

Step2: Here $T^0 = 40$ days. Then, for $T_1 \geq 30$, $T_1^0 = 31.54$ days and $C^1_{30}(T_1^0, T^0) = \$111.62$.

For $T_1 < 30$, $T_1^0 = 27.14$ days and $C^2_{30}(T_1^0, T^0) = \$109.47$.

Hence, $C_{30}(T_1^0, T^0) = \$109.47$ with $T_1^0 = 27.14$ days and $T^0 = 40$ days.

Now consider $M = 15$ days.

For $T_1 \geq 15$, $T_{opt} = 28$ days, $T_{1opt} = 21$ days, and $C^1_{15}(T_{1opt}, T_{opt}) = \$115.89$

For $T_1 < 15$, $T_{\text{opt}} = 22.35$ days, $T_{1opt} = 14.91$ days, and $C^2_{15}(T_{1opt}, T_{opt}) = \$122.20$.

Therefore, optimal T and T_1 minimizing $C_{15}(T_1, T)$ are $T^{**} = 28$ days, $T_1^{**} = 21$ days, and minimum cost in this case is $C_{15}(T_1^*, T^*) = \$115.89$.

Since $C_{15}(T_1^*, T^*) = \$115.89 > C_{30}(T_1^0, T^0) = \$109.47$, $T^0 = 40$ days, $T_1^0 = 27.14$ days are optimal values of T and T_1 respectively, and the minimum cost per day is \$109.47.

EXAMPLE 2

Let K= \$10.00 per order, h = \$15.00 per unit, P = \$100.00 per unit, s = \$25.00 per unit, $\alpha = 250$ units, $I_r = 0.15$, $I_e = 0.10$ and

$$M = \begin{cases} 30 \text{ days if } 0 < q < 10000 \\ 45 \text{ days if } q \geq 10000 \end{cases}$$

Step1: First let $M = 45$ days.

For $T_1 \geq 45$, $T_{opt} = 54$ days, $T_{1opt} = 45$ days and $C^1_{45}(T_{1opt}, T_{opt}) = \$144.65$.

For $T_1 < 45$, $T_{opt} = 45$ days, $T_{1opt} = 28.64$ days and $C^2_{45}(T_{1opt}, T_{opt}) = \$91.93$.

Hence, optimal T and T_1 minimizing $C_{45}(T_1, T)$ are $T^* = 45$ days, $T_1^* = 28.64$ days and minimum cost is $C_{45}(T_1^*, T^*) = \$ 87.41$.

Since $\alpha T^* = 11250 > q_0 = 10000$, $T^* = 45$ days, $T_1^* = 28.64$ days are optimal with minimum cost per day \$91.93.

5. Conclusion

The paper studies an inventory problem, where the shortage is completely backlogged and the permissible delay in payment depends on the order quantity. An algorithm is suggested to find the optimal ordering policy, which helps the inventory manager to decide whether it would be worthwhile to take advantage of a longer credit period for repaying the supplier by ordering a larger amount of the commodity. The paper, however, considers only one break in the delay period. A natural extension of the model would be to study the case of N breaks in the permissible delay period, i.e. to assume

$$M = M_i, \text{ if } q_{i-1} \le q < q_i \text{ , } i = 1,2,...,N,$$

where $M_1 < M_2 < ... < M_N$, and $q_0 = 0 < q_1 < q_2 < ... < q_N = \infty$.
It would also be interesting to study the problem discussed in the paper for a deteriorating item.

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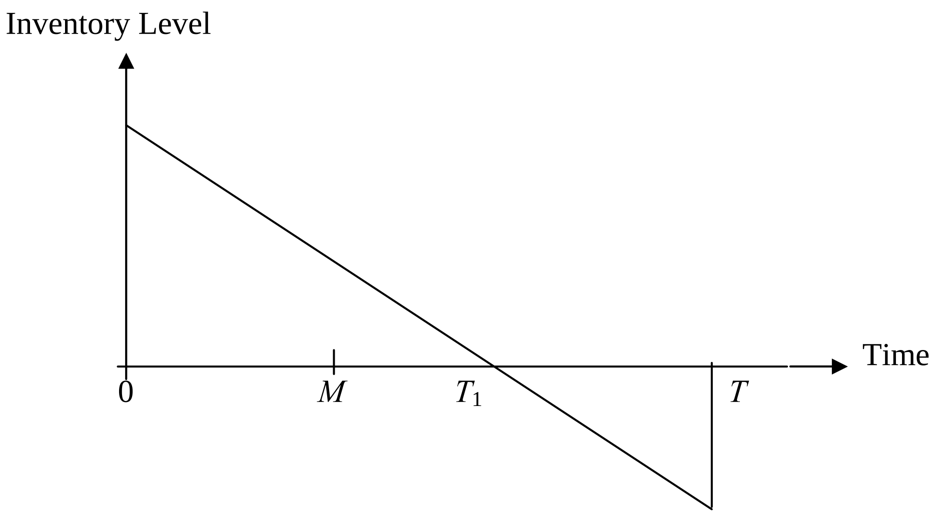
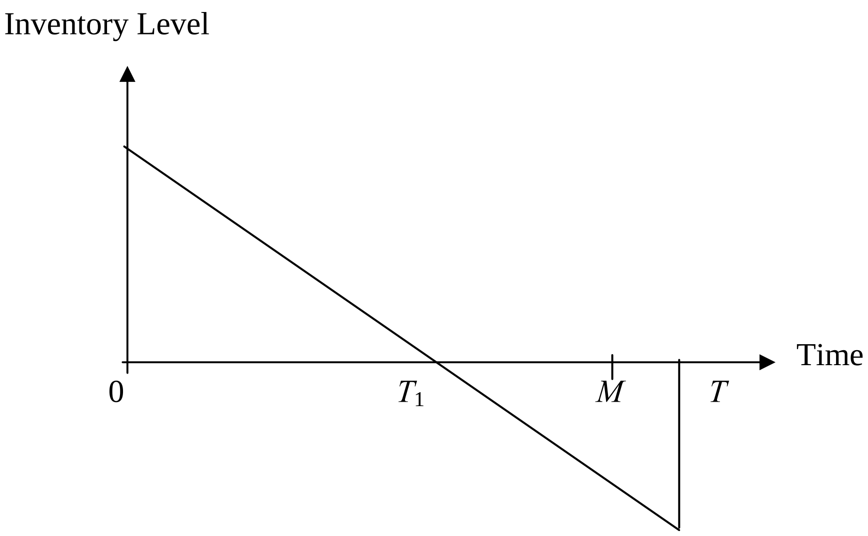
FIGURE 1



CASE 1: $M \leq T_1$

FIGURE 2



CASE 2: $M > T_1$