Post-optimality Analysis in Bounded Variables Problem

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Abstract

This paper deals with dual simplex algorithm and sensitivity -analysis (or post-optimality analysis) in linear programming with bounded variables .In sensitivity analysis, change in coefficient matrix A, deletion of a variable and deletion of a constraint have been discussed. Numerical illustration is also included in support of theory.

Keywords: Dual-simplex algorithm, sensitivity-analysis, linear programming.

1. Introduction

Linear programming with bounded variables have been studied by many authors (Dantzig, 1963), (Turnvec,1968), (Duguay,1972), (Finke,1979), (Cavatal,1983), (Ho,1991), (Hussain, 2000). In1954, Dantzig developed the method for solving linear programming with upper bound restrictions on the variables. In 1972, Duguay et al. studied linear programming with relative bounded variables. Later on, various methods like revised simplex algorithm, modified decomposition algorithm have been developed by various authors (Murty,1976), (Ho,1991). This paper is concerned with sensitivity analysis for linear programming with bounded variables. Sensitivity analysis (also called post optimality analysis) is the study of the behaviour of the optimal solution with respect to changes in the input parameters of the original optimization problem. It is often as important as solving the original problem itself, partly because in real life linear programs, the data are rarely exact. They are often estimates, subject to measurement errors, or are simply uncertain. For example, prices may change daily or even hourly, and one may not know for sure how much of a resource one will have in a given planning horizon. For these reasons, one is often interested in how the optimal solution would change if the data in the linear programming is perturbed in various ways. Sensitivity analysis for linear programs with non-negative variables have been discussed by (Ward, Wendell,1990), (Pu Cai, Jin-Yi,1997). In 2001, Yildirim et al. discussed sensitivity analysis in linear programming and semidefinite programming with non-negative variables using interior point method. While solving linear programming with bounded variables using sensitivity analysis, sometime we get a solution, which is optimal but not feasible. Then dual simplex algorithm is the method of choice when linear programs have to be reoptimized when data in program is perturbed. In 1958, Wagner gave dual simplex algorithm for solving linear programming

$$\max Z = C^T X$$

$$\text{subject to } AX = b, \text{ and } 0 \le X \le U.$$

This algorithm involves the replacement, after each cycle of the algorithm, any variable x_i which has exceeded its upper bound by its complementary variable $x_i^{'}$, where $x_i + x_i^{'} = u_i$. This involves only a set of sign changes, relabelling, and a trivial numerical substitution and returns the tableau to standard form.

Our paper contains four sections. In section 2, we discuss the dual simplex algorithm for bounded variable problem. In section 3, analysis of perturbation in coefficient matrix A, deletion of a variable and deletion of a constraint is given. In section 4, numerical illustration is given.

We first briefly discuss the basic results used in linear programming with bounded variables which are discussed in (Murty, 1976),(Swarup,2000).

Consider the following linear programming

$$(P) \quad \max Z = C^T X$$

$\text{subject to } AX = b, \text{ and } L \le X \le U,$ where $X, C, L, U \in R^n$; $b \in R^m$; α, β are scalars and A is an $m \times n \, (m \le n)$ matrix. L and U are lower and upper bounds on the decision variables.

We will make the following two assumptions.

Assumption 1. The coefficient matrix A has full row rank i.e. $\rho(A) = m$

Assumption 2. L is a non-negative $(n \times 1)$ vector.

Let $A = [B \;\; N_1 \;\; N_2]$, $X = [X_B \;\; X_{N_1} \;\; X_{N_2}]^T$, where B is a basis matrix, N_1, N_2 are the sub

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matrices of A associated with non-basic variables which are at their lower and upper bounds respectively. X_B, X_{N_1}, X_{N_2} are the variables associated with B, N_1, N_2 respectively.

Consider AX=b,

$$\Rightarrow BX_B + N_1 X_{N_1} + N_2 X_{N_2} = b,$$

$$X_B + B^{-1}N_1 X_{N_1} + B^{-1}N_2 X_{N_2} = B^{-1}b,$$

$$\Rightarrow X_B + \sum_{a_j \in N_1} Y_j x_j + \sum_{a_j \in N_2} Y_j x_j = B^{-1}b. \qquad (1)$$

Currently, $X_B = X_B^o$ (say)=current value of basic variables;

$x_j = l_j, \forall a_j \in N_1;\ x_j = u_j, \forall a_j \in N_2.$ So (1) implies that

$$X_B^o + \sum_{a_j \in N_1} Y_j l_j + \sum_{a_j \in N_2} Y_j u_j = B^{-1}b, \qquad (2)$$

$$\text{i.e. } x_{B_i}^o + \sum_{a_j \in N_1} y_{ij} l_j + \sum_{a_j \in N_2} y_{ij} u_j = (B^{-1}b)_i,\quad \forall\ i = 1,2,\ldots,m.$$

$$x_{B_i}^o = (B^{-1}b)_i - \sum_{a_j \in N_1} y_{ij} l_j - \sum_{a_j \in N_2} y_{ij} u_j,\quad \forall\ i = 1,2,\ldots,m.$$

Equation (1) and (2) imply that

$$\left\{ \begin{array}{l} X_B + \sum_{a_j \in N_1} Y_j (x_j - l_j) \\ + \sum_{a_j \in N_2} Y_j (x_j - u_j) \end{array} \right\} = X_B^o,$$

$$\text{i.e.} \left\{ \begin{array}{l} x_{B_i} + \sum_{a_j \in N_1} y_{ij} (x_j - l_j) \\ + \sum_{a_j \in N_2} y_{ij} (x_j - u_j) \end{array} \right\} = x_{B_i}^o,\quad \forall\ i = 1,2,\ldots,m. \qquad (3)$$

Equation (3) will be used in reading and appending constraints in linear programming with bounded variables.

Optimality Criterion:

$$Z(X) = C^T X$$

$$= C_B^T X_B + C_{N_1}^T X_{N_1} + C_{N_2}^T X_{N_2}$$

$$= \left\{ \begin{array}{l} C_B^T (B^{-1}b) - \sum_{a_j \in N_1} (Z_j - C_j) x_j \\ - \sum_{a_j \in N_2} (Z_j - C_j) \end{array} \right\}$$

[Using (1)] (4)

where $Z_j = \sum_{i=1}^{m} C_{B_i} y_{ij}$.

$$Z^o = \text{Current value of } Z$$

$$= \left\{ \begin{array}{l} C_B^T (B^{-1}b) - \sum_{a_j \in N_1} (Z_j - C_j) l_j \\ - \sum_{a_j \in N_2} (Z_j - C_j) u_j \end{array} \right\}. \qquad (5)$$

Result 1. A basic feasible solution $X = (X_B\ X_{N_1}\ X_{N_2})^T$ will be an optimal basic feasible solution of problem (P) if

$$Z_j - C_j \geq 0 \quad \forall\ a_j \in N_1,$$

$$Z_j - C_j \leq 0 \quad \forall\ a_j \in N_2,$$

$$\text{and} \quad Z_j - C_j = 0 \text{ for all basic variables.}$$

Note: Such solution as defined in Result 1, is called primal as well as dual feasible. If X is optimal solution but not a feasible solution of problem (P), then it is called dual feasible solution.

Let some non-basic variable x_j undergoes change say Δ_j and currently $x_j = l_j$. Let $\hat{x}_j$ = New value of $x_j = l_j + \Delta_j$, then

$$\hat{X}_B = X_B^o - Y_j \Delta_j,$$

$$\text{i.e. } \hat{x}_{B_i} = x_{B_i}^o - y_{ij}\Delta_j,\ \forall\ i = 1,2,\ldots,m. \quad (6)$$

Similarly when a non-basic variable x_j which is currently at its upper bound undergoes change i.e. $x_j = u_j \rightarrow \hat{x}_j = u_j - \Delta_j$, then

$$\hat{x}_{B_i} = x^o_{B_i} + y_{ij}\Delta_j, \forall \ i = 1,2,\ldots,m. \qquad (7)$$

2. Dual Simplex Algorithm for bounded variables

The dual simplex algorithm starts with a dual feasible basis but primal infeasible and walks to a terminal basis by moving along adjacent dual feasible basis. At each pivot step, this algorithm tries to reduce primal infeasibility while retaining dual feasibility.

Let B be a known dual feasible basis and $X^o_B = (x^o_{B_1}, x^o_{B_2}, \cdots, x^o_{B_m})$ be the associated basic vector. Suppose r^{th} basic variable $x^o_{B_r}$ is not within its bounds, so we depart this basic variable and enter some non-basic variable say $a_k \notin B$.

There are two possibilities: Either $x^o_{B_r}$ is below its lower bound or above its upper bound.

Case I: If $x^o_{B_r}$ is below its lower bound.

While applying dual simplex iteration in this case, our aim is to increase $x^o_{B_r}$ till it attains its lower bound. Again there are two possibilities

(i) $a_k \in N_1$ (ii) $a_k \in N_2$.

(i) Let $a_k \in N_1$, which is currently non-basic and at its lower bound with $Z_k - C_k \geq 0$, is selected for replacing b_r, where $B = (b_1\, b_2 \cdots b_m)$.

Let $\hat{x}_k = l_k + \Delta_k$, where Δ_k in non-negative and determined by

$$\hat{x}_{B_r} = x^o_{B_r} - y_{rk}\Delta_k = l_r, \text{ where } y_{rk} \text{ is the pivot element.}$$

Note that $y_{rk} \leq 0$. Since $\Delta_k > 0$ and for increasing x_{B_r}, y_{rk} should be negative.

$$\Rightarrow \Delta_k = \frac{l_r - x^o_{B_r}}{-y_{rk}}$$

When a_k enters and b_r departs, then

$$\hat{Z}_j - C_j = (Z_j - C_j) - \frac{y_{rj}}{y_{rk}}(Z_k - C_k), \quad \forall \ a_j \notin B. \qquad (8)$$

For maintaining optimality,

$$\hat{Z}_j - C_j \geq 0, \ \forall a_j \in N_1 \text{ and}$$

$$\hat{Z}_j - C_j \leq 0, \ \forall a_j \in N_2.$$

For $y_{rj} < 0, \hat{Z}_j - C_j \geq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \geq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_1. \qquad (9)$$

Clearly from (8), for $y_{rj} \leq 0, \hat{Z}_j - C_j \leq 0$, for $a_j \in N_2$.

For $y_{rj} > 0, \hat{Z}_j - C_j \leq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \geq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_2. \qquad (10)$$

Relations (9) and (10) imply that

$$\frac{Z_k - C_k}{y_{rk}} = \max\left\{\begin{array}{l}\left(\dfrac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} < 0\right), \\ \left(\dfrac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} > 0\right)\end{array}\right\}. \qquad (11)$$

(ii) Let $a_k \in N_2$, which is currently non-basic and at its upper bound with $(Z_k - C_k) \leq 0$ is selected for replacing b_r.

Let $\hat{x}_k = u_k - \Delta_k$, where Δ_k in non-negative and determined by

$\hat{x}_{B_r} = x^o_{B_r} + y_{rk}\Delta_k = l_r$, where y_{rk} is the pivot element.

Note that $y_{rk} \geq 0$. Since $\Delta_k > 0$ and for increasing x_{B_r}, y_{rk} should be positive.

$$\Rightarrow \Delta_k = \frac{l_r - x^o_{B_r}}{y_{rk}}$$

For maintaining optimality,

$$\hat{Z}_j - C_j \geq 0, \ \forall a_j \in N_1 \text{ and}$$

$$\hat{Z}_j - C_j \leq 0, \ \forall a_j \in N_2.$$

Form (8), for $a_j \in N_1, \hat{Z}_j - C_j \geq 0$, for $y_{rj} \geq 0$. For $y_{rj} < 0, \hat{Z}_j - C_j \geq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \geq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_1. \quad (12)$$

Form(8), for $a_j \in N_2, \hat{Z}_j - C_j \leq 0$, for $y_{rj} \leq 0$. For $y_{rj} > 0, \hat{Z}_j - C_j \leq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \geq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_2. \quad (13)$$

Relations (12) and (13) imply that $\frac{Z_k - C_k}{y_{rk}} =$

$$\max\left\{ \begin{aligned} &\left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} < 0\right), \\ &\left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} > 0\right) \end{aligned} \right\}. \quad (14)$$

which is same as (11).

Case II: If $x^o_{B_r}$ is above its upper bound.

While applying dual simplex iteration in this case, our aim is to decrease $x^o_{B_r}$ till it attains its upper bound. Again there are two possibilities

(i) $a_k \in N_1$ (ii) $a_k \in N_2$.

(i) Let $a_k \in N_1$, which is currently non-basic and at its lower bound with $Z_k - C_k \geq 0$, is selected for replacing b_r, where $B = (b_1 \, b_2 \cdots b_m)$.

Let $\hat{x}_k = l_k + \Delta_k$, where Δ_k in non-negative and determined by

$\hat{x}_{B_r} = x^o_{B_r} - y_{rk}\Delta_k = u_r$, where y_{rk} is the pivot element.

Note that $y_{rk} \geq 0$. Since $\Delta_k > 0$ and for decreasing x_{B_r}, y_{rk} should be positive.

$$\Rightarrow \Delta_k = \frac{x^o_{B_r} - u_r}{y_{rk}}$$

As in Case I, for maintaining optimality,

$$\hat{Z}_j - C_j \geq 0, \ \forall a_j \in N_1 \text{ and}$$

$$\hat{Z}_j - C_j \leq 0, \ \forall a_j \in N_2.$$

Form (8), for $a_j \in N_1, \hat{Z}_j - C_j \geq 0$, for $y_{rj} \leq 0$. For $y_{rj} > 0, \hat{Z}_j - C_j \geq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \leq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_1. \quad (15)$$

Form (8), for $a_j \in N_2, \hat{Z}_j - C_j \leq 0$, for $y_{rj} \geq 0$. For $y_{rj} < 0, \hat{Z}_j - C_j \leq 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \leq \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_2. \quad (16)$$

Relations (15) and (16) imply that

$$\frac{Z_k - C_k}{y_{rk}} = \min\left\{\begin{array}{l}\left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} > 0\right),\\ \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} < 0\right)\end{array}\right\}. \quad (17)$$

(ii) Let $a_k \in N_2$, which is currently non-basic and at its upper bound with $(Z_k - C_k) \le 0$ is selected for replacing b_r.

Let $\hat{x}_k = u_k - \Delta_k$, where Δ_k in non-negative and determined by

$$\hat{x}_{B_r} = x^o_{B_r} + y_{rk}\Delta_k = u_r, \text{ where } y_{rk} \text{ is the pivot element.}$$

Note that $y_{rk} \le 0$. Since $\Delta_k > 0$ and for decreasing x_{B_r}, y_{rk} should be negative.

$$\Rightarrow \Delta_k = \frac{x^o_{B_r} - u_r}{-y_{rk}}$$

For maintaining optimality,

$$\hat{Z}_j - C_j \ge 0, \ \forall a_j \in N_1 \text{ and}$$

$$\hat{Z}_j - C_j \le 0, \ \forall a_j \in N_2.$$

Form (8),for $a_j \in N_1, \hat{Z}_j - C_j \ge 0,$ for $y_{rj} \le 0.$ For $y_{rj} > 0, \hat{Z}_j - C_j \ge 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \le \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_1. \quad (18)$$

Form (8), for $a_j \in N_2, \hat{Z}_j - C_j \le 0,$ for $y_{rj} \ge 0.$ For $y_{rj} < 0, \hat{Z}_j - C_j \le 0$ if

$$\frac{Z_k - C_k}{y_{rk}} \le \frac{Z_j - C_j}{y_{rj}} \ \forall a_j \in N_2. \quad (19)$$

Relations (18) and (19) imply that

$$\frac{Z_k - C_k}{y_{rk}} = \min\left\{\begin{array}{l}\left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} > 0\right),\\ \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} < 0\right)\end{array}\right\}, (20)$$

which is same as (17).

Result 2 (Primal infeasibility criterion) : The original problem (P) is infeasible if corresponding to a dual feasible basis B, there exists an i such that either

(I) $x_{B_i} = \bar{b}_i < l_{B_i}$ and $y_{ij} \ge 0 \ \forall \ a_j \in N_1$ and $y_{ij} \le 0 \, \forall \, a_j \in N_2.$ or

(II) $x_{B_i} = \bar{b}_i > u_{B_i}$ and $y_{ij} \le 0 \ \forall \ a_j \in N_1$ and $y_{ij} \ge 0 \, \forall \, a_j \in N_2.$

Proof: Suppose condition (I) is satisfied in the i^{th} row corresponding to dual feasible basis B. This row corresponds to the constraint

$$\left\{\begin{array}{l} x_{B_i} + \sum_{a_j \in N_1} y_{ij}(x_j - l_j) + \\ \sum_{a_j \in N_2} y_{ij}(x_j - u_j) \end{array}\right\} = \bar{b}_i \quad (21)$$

Since (21) is a linear combination of the original constraints in the problem (P). Hence, every feasible solution of (P) must satisfy (21). However $\bar{b}_i < l_{B_i}$ and $y_{ij} \ge 0 \ \forall \ a_j \in N_1$ and $y_{ij} \le 0 \, \forall \, a_j \in N_2$, (21) can't be satisfied by any X such that $L \le X \le U$. Hence the primal problem is infeasible. Similarly, primal infeasibility can be proved when condition (II) is satisfied.

Result 3 (Dual simplex entering criterion): If some $x_{B_r} (< l_{B_r})$ is chosen to leave the basis then the variable x_k enters the basis if

$$\frac{Z_k - C_k}{y_{rk}} = \max\left\{ \begin{array}{l} \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} < 0 \right), \\ \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} > 0 \right) \end{array} \right\},$$

and if some $x_{B_r} (> u_{B_r})$ is chosen to leave the basis then the variable x_k enters the basis if

$$\frac{Z_k - C_k}{y_{rk}} = \min\left\{ \begin{array}{l} \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_1, y_{rj} > 0 \right), \\ \left(\frac{Z_j - C_j}{y_{rj}} \mid a_j \in N_2, y_{rj} < 0 \right) \end{array} \right\}.$$

If any of the criterion is not applicable, then there exists no feasible solution to the problem (P) (by *Result 2*).

Steps of the Algorithm

Step 1. Convert the minimization problem into maximization if it is minimization form. Convert the $\geq$ type inequalities, representing the constraints of the given linear programming, if any, into those of $\leq$ type. Call this problem as (P).

Step 2. Introduce slack variables in the constraints of the given problem and obtain an initial basic dual feasible solution and consider the corresponding starting dual simplex table.

Step 3. Test the nature of $(Z_j - C_j)$ in the starting simplex table.

(a) If $l_j \leq x_l \leq u_j \; \forall \; j = 1,2,\ldots,n$ and $Z_j - C_j \geq 0 \; \forall \; a_j \in N_1$ and $Z_j - C_j \leq 0 \; \forall \; a_j \in N_2$,

then an optimal basic feasible solution of problem (P) has been obtained.

(b) If $Z_j - C_j \geq 0 \; \forall \; a_j \in N_1$ and

$Z_j - C_j \leq 0 \; \forall \; a_j \in N_2$ and at least one basic variable say x_{B_i} is not within its bounds, then go to step $4(a)$ or $4(b)$ accordingly as $x_{B_i} < l_i$ or $x_{B_i} > u_i$.

Step 4(a) Select that basic variable x_{B_i} for which $| x_{B_i} - l_i |$ is maximum. Let x_{B_k} be such that $| x_{B_k} - l_k |$ is maximum so that Y_k leaves the basis. Go to step $5(a)$.

(b) Select that basic variable x_{B_i} for which $| x_{B_i} - u_i |$ is maximum. Let x_{B_k} be such that $| x_{B_k} - u_k |$ is maximum so that Y_k leaves the basis. Go to step $5(b)$.

Step 5(a) Test the nature of $y_{kj},\; j = 1,2,\ldots,n.$

(i) If $y_{kj} \geq 0 \; \forall \; a_j \in N_1$ and $y_{kj} \leq 0 \; \forall \; a_j \in N_2$, there does not exist any feasible solution to the given problem (by *Result 2*).

(ii) If at least one y_{kj} is negative for some $a_j \in N_1$ or y_{kj} is positive for some $a_j \in N_2$, compute the replacement ratios

$$\left\{ \begin{array}{l} \left(\frac{Z_j - C_j}{y_{kj}} \mid a_j \in N_1, y_{kj} < 0 \right), \\ \left(\frac{Z_j - C_j}{y_{kj}} \mid a_j \in N_2, y_{kj} > 0 \right) \end{array} \right\}$$

and choose the maximum of these. The corresponding column vector, say Y_r, then enters the basis.

(b) Test the nature of $y_{kj},\; j = 1,2,\ldots,n.$

(i) If $y_{kj} \leq 0 \; \forall \; a_j \in N_1$ and $y_{kj} \geq 0 \; \forall \; a_j \in N_2$, there does not exist any feasible solution to the given problem (by *Result 2*).

(ii) If at least one y_{kj} is positive for some $a_j \in N_1$ or y_{kj} is negative for some $a_j \in N_2$, compute the replacement ratios

$$\left\{ \begin{array}{l} \left(\frac{Z_j - C_j}{y_{kj}} \mid a_j \in N_1, y_{kj} > 0 \right), \\ \left(\frac{Z_j - C_j}{y_{kj}} \mid a_j \in N_2, y_{kj} < 0 \right) \end{array} \right\}$$

and choose the minimum of these. The corresponding column vector, say Y_r, enters the basis.

Step 6. Test the new iterated dual simplex table for dual optimality. Repeat the method until either an optimum feasible solution has been obtained (in a finite number of steps) or there is an indication of non-existence of a primal feasible solution.

Remark 1: If upper bounds of all the decision variables are finite then primal problem is bounded. If not so, then the dual problem is known to be feasible, the primal problem cannot be unbounded, by weak duality theorem. The algorithm discussed here will terminate with a basis that satisfies either the optimality criterion or primal infeasibility criterion.

3. Sensitivity Analysis

Consider following three changes in data, namely

(i) Deletion of a variable,

(ii) Deletion of a constraint,

(iii) Change in the coefficient matrix A.

Deletion of a Variable

Case (a): Deletion of a non-basic variable

When a non-basic variable say, x_k is dropped, then basis and $Z_j - C_j$ will not change, only X_B and Z will undergo change, Y_k will be taken away.

Let $a_k \notin B$ be dropped. There are two possibilities either $a_k \in N_1$ or $a_k \in N_2$.

Case (i): $a_k \in N_1$, then

$X_B' = X_B + Y_k\, l_k$ and $Z' = Z + (Z_k - C_k) l_k$.

In this case optimality is maintained but feasibility may be hampered. If $L_B \le X_B' \le U_B$, then new solution is optimal as well as feasible. If X_B' is not feasible then apply dual simplex algorithm and proceed.

Case (ii): $a_k \in N_2$

Parallel to case (i).

Case (b): Deletion of a basic variable

Deletion of a basic variable may affect the optimality as well as feasibility. For deletion of x_{B_i}, we make x_{B_i} a non-basic, give it a high negative cost $-M$ ($+M$ in minimization case) in optimal table of the problem (P) and also change its bounds $l_{B_i} = 0, u_{B_i} = \infty$. Calculate revised values of Z and $Z_j - C_j$. And

$$Z_j' - C_j' = C_B' Y_j - C_j$$
$$= \sum_{r=1, r\ne i}^{m} C_{B_r} y_{rj} + M\, y_{ij} - C_j$$
$$= (Z_j - C_j) + (M - C_{B_i})\, y_{ij}$$

$$Z' = \begin{bmatrix} C_B'(B^{-1}b) - \sum_{a_j \in N_1} (Z_j' - C_j')l_j - \\ \sum_{a_j \in N_2} (Z_j' - C_j')u_j \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1, r\ne i}^{m} C_{B_r}(B^{-1}b)_r - M(B^{-1}b)_i - \\ \sum_{a_j \in N_1} [(Z_j - C_j) + (M - C_{B_i})y_{ij}]l_j - \\ \sum_{a_j \in N_2} [(Z_j - C_j) + (M - C_{B_i})y_{ij}]u_j + \\ C_{B_i}\, x_{B_i} - C_{B_i}\, x_{B_i} \end{bmatrix}$$

$$= Z^o - (M + C_{B_i})\, x_{B_i}.$$

Now, x_{B_i} serves as an artificial variable. While making these changes, only optimality can hamper. If optimality is hampered, then apply simplex algorithm and find optimal solution. In the optimal table check, whether x_{B_i} is basic or non-basic.

If x_{B_i} is non-basic then it will be at its lower bound and now delete its column, no change in objective function value and basic variables. If x_{B_i} is basic and not replaceable then the problem will be infeasible [8,Chapter 2,p.43-44)], otherwise replace it and proceed as discussed above.

(ii) Deletion of a constraint

Case(a) Deletion of inactive constraint

An inactive constraint is that one which is satisfied as strict inequality. So its corresponding slack or surplus variable would be basic and at non-zero level. Suppose we want to delete i^{th} constraint which is inactive. Then delete the row and column of the slack/surplus variable corresponding to i^{th}

constraint. There will be no change in X_B, Z and $Z_j - C_j$.

Case(b) Deletion of an active constraint(Binding constraint)

A constraint, which is satisfied as an equation, is called an active constraint. Let i^{th} constraint is active and we want to delete it. For this, we make this constraint inactive and then proceed as in case(a). To make it inactive its slack/surplus must be introduced into basis at positive level. So give slack/surplus high positive cost $+M$ ($-M$ in minimization case) and calculate $Z_j^{'} - C_j^{'}$ for this slack/surplus variable and enter slack/surplus variable into basis at next iteration. This makes the constraint inactive. Cut the row and column of corresponding slack/surplus variable.

Note: Let x_{S_i} be the slack variable in i^{th} constraint, which is active in optimal table. As $x_{S_i} \geq 0$ and has no finite upper bound. So, if x_{S_i} is non-basic, then it will be at its lower bound only and when $C_{S_i} \to M, Z_{S_i}^{'} - C_{S_i}^{'} = C_B Y_{S_i} - M < 0$, so it will always enter the basis and make constraint inactive.

(iii) Change in A

Change in A is equivalent to deletion of one constraint and addition of another. This change in A can be handled by applying the results for addition and deletion of constraints as discussed earlier. Now we are interested in finding the range by which each component of any activity vector can be changed one at a time so that same solution remains optimal basic feasible solution of new problem.

Let $B = (b_1 \ b_2 \cdots b_m)$ be the optimal feasible basis for the original problem and $A = [a_1 \ a_2 \ \cdots a_s \cdots a_n]$ and a_s undergoes change.

There are two possibilities:

Case(I) $a_s \notin B$ Case(II) $a_s \in B$.

Case(I) $a_s \notin B$.

If $a_s \notin B$, then either $a_s \in N_1$ or $a_s \in N_2$.

If $a_s \in N_1$, then $x_s = l_s$.

$$\text{Let } a_s = \begin{pmatrix} a_{1s} \\ a_{2s} \\ \vdots \\ a_{ks} \\ \vdots \\ a_{ms} \end{pmatrix} \Rightarrow a_s^{'} = \begin{pmatrix} a_{1s} \\ a_{2s} \\ \vdots \\ a_{ks} + \delta a_{ks} \\ \vdots \\ a_{ms} \end{pmatrix}.$$

$$Y_s^{'} = B^{-1} a_s^{'} = B^{-1} \left[a_s + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \delta a_{ks} \\ \vdots \\ 0 \end{pmatrix} \right] = Y_s + \overleftarrow{\beta}_k \delta a_{ks},$$

where $B^{-1} = (\overleftarrow{\beta}_1 \overleftarrow{\beta}_2 \cdots \overleftarrow{\beta}_k \cdots \overleftarrow{\beta}_m)$.

$$X_B^{'} = X_B - \overleftarrow{\beta}_k \delta a_{ks} l_s$$

$$\Leftrightarrow x_{B_i}^{'} = x_{B_i} - \beta_{ik} \delta a_{ks} l_s \quad \forall\, i = 1, 2, \cdots, m.$$

$$Z_j^{'} - C_j^{'} = Z_j - C_j \quad \forall \ a_j \in (N_1 \bigcup N_2) \setminus \{s\}$$

and

$$\begin{aligned} Z_s^{'} - C_s^{'} &= C_B Y_s^{'} - C_s \\ &= C_B (Y_s + \overleftarrow{\beta}_k \delta a_{ks}) - C_s \\ &= (Z_s - C_s) + C_B \overleftarrow{\beta}_k \delta a_{ks}. \end{aligned}$$

$$Z^{'} = \begin{bmatrix} C_B(B^{-1}b) - \sum_{a_j \in N_1} (Z_j^{'} - C_j^{'}) l_j - \\ \sum_{a_j \in N_2} (Z_j^{'} - C_j^{'}) u_j \end{bmatrix}$$

$$= \begin{bmatrix} C_B(B^{-1}b) - \sum_{a_j \in N_1,\, j \neq s} (Z_j - C_j) l_j \\ - \sum_{a_j \in N_2} (Z_j - C_j) u_j \\ -[(Z_s - C_s) + C_B \overleftarrow{\beta}_k \delta a_{ks}] l_s \end{bmatrix}$$

$$= Z^o - C_B \overleftarrow{\beta}_k \delta a_{ks} l_s.$$

So change in $a_s \in N_1$ affects both optimality as well as feasibility.

Range of δa_{ks} so that optimality is maintained.

For solution $X_B^{'}$ to be optimal,

$$Z_j^{'} - C_j^{'} \geq 0 \; \forall \, a_j \in N_1 \text{ and } Z_j^{'} - C_j^{'} \leq 0 \; \forall \, a_j \in N_2 .$$

$$\text{i.}e. \; (Z_s - C_s) + C_B \overleftarrow{\beta}_k \delta a_{ks} \geq 0 \text{ for } a_s \in N_1,$$

$$[\text{Since } Z_j^{'} - C_j = Z_j - C_j, \forall \; a_j \in (N_1 \bigcup N_2) \setminus \{s\}],$$

$$C_B \overleftarrow{\beta}_k \delta a_{ks} \geq -(Z_s - C_s),$$

$$\delta a_{ks} \geq -\frac{(Z_s - C_s)}{C_B \overleftarrow{\beta}_k} \text{ if } C_B \overleftarrow{\beta}_k > 0,$$

$$\delta a_{ks} \leq -\frac{(Z_s - C_s)}{C_B \overleftarrow{\beta}_k} \text{ if } C_B \overleftarrow{\beta}_k < 0,$$

$$\text{i.}e. \left\{ -\frac{(Z_s - C_s)}{C_B \overleftarrow{\beta}_k} \,|\, C_B \overleftarrow{\beta}_k > 0 \right\} \leq \delta a_{ks} \leq \left\{ -\frac{(Z_s - C_s)}{C_B \overleftarrow{\beta}_k} \,|\, C_B \overleftarrow{\beta}_k < 0 \right\}. \quad (22)$$

Range of δa_{ks} so that feasibility is maintained.

For feasibility, $L_B \leq X_B^{'} \leq U_B$

$$\text{i.}e. \; l_{B_i} \leq x_{B_i} - \beta_{ik} \delta a_{ks} l_s \leq u_{B_i} \quad \forall \, i = 1, 2, \cdots, m. \quad (23)$$

If $\beta_{ik} l_s > 0,$ then (23) hold if

$$\max_i \left\{ \frac{u_{B_i} - x_{B_i}}{-\beta_{ik} l_s} \,|\, \beta_{ik} l_s > 0 \right\} \leq \delta a_{ks} \leq \min_i \left\{ \frac{x_{B_i} - l_{B_i}}{\beta_{ik} l_s} \,|\, \beta_{ik} l_s > 0 \right\}. \quad (24)$$

If $\beta_{ik} l_s < 0,$ then (23) hold if

$$\max_i \left\{ \frac{x_{B_i} - l_{B_i}}{\beta_{ik} l_s} \,|\, \beta_{ik} l_s < 0 \right\} \leq \delta a_{ks} \leq \min_i \left\{ \frac{u_{B_i} - x_{B_i}}{-\beta_{ik} l_s} \,|\, \beta_{ik} l_s < 0 \right\} \quad (25)$$

Relations (24) and (25) together imply that for maintaining feasibility δa_{ks} should lie in

$$\max_i \left\{ \begin{array}{l} \left(\frac{x_{B_i} - l_{B_i}}{\beta_{ik} l_s} \,|\, \beta_{ik} l_s < 0 \right), \\ \left(\frac{u_{B_i} - x_{B_i}}{-\beta_{ik} l_s} \,|\, \beta_{ik} l_s > 0 \right) \end{array} \right\} \leq \delta a_{ks} \leq \min_i \left\{ \begin{array}{l} \left(\frac{u_{B_i} - x_{B_i}}{-\beta_{ik} l_s} \,|\, \beta_{ik} l_s < 0 \right), \\ \left(\frac{x_{B_i} - l_{B_i}}{\beta_{ik} l_s} \,|\, \beta_{ik} l_s > 0 \right) \end{array} \right\}. \quad (26)$$

Similarly, if $a_s \in N_2$ undergoes change parallel results will be obtained by replacing l_s by u_s. Range of δa_{ks} so that optimality as well as feasibility both are maintained is obtained by taking union of (22) and (26).

Remark 2: Solution of the problem when some $a_s \in N_1$ undergoes change

If only optimality is hampered, apply simplex algorithm and solve. If only feasibility is hampered, apply dual simplex algorithm and solve. If both optimality as well as feasibility are hampered, then x_s which is currently at lower bound, set at its upper bound $x_s = u_s$ and calculate

$$\overline{X}_B = X_B^{'} - Y_j^{'} u_s,$$

$$\text{and } \overline{Z} = Z^{'} - (Z_s^{'} - C_s^{'}) u_s .$$

All other relative cost coefficients, basis and Y_j remain unaltered during this change. Now x_s is at its upper bound and $Z_s - C_s < 0$. So this solution is optimal but need not be feasible. If $\overline{X}_B$ is feasible, then it is optimal basic feasible solution, otherwise apply dual simplex algorithm and solve. Similarly we can solve for $a_s \in N_2$.

Case(II) $a_s \in B$ (say $a_s = b_r$)

Let $X = (X_B \;\; L_{N_1} \;\; L_{N_2})$ be the optimal basic feasible solution of the original problem (P) corresponding to optimal basis B. When $a_s \to a_s^{'}$, let $B \to B^{'}$.

Let $B = (b_1 \;\; b_2 \cdots b_k \cdots b_m)$ and $B^{-1} = (\overleftarrow{\beta}_1 \; \overleftarrow{\beta}_2 \cdots \overleftarrow{\beta}_k \cdots \overleftarrow{\beta}_m)$.

Calculation of $(B^{'})^{-1}$

$$
\begin{aligned}
B^{-1}a_s' &= B^{-1}(a_s + [0\cdots\delta a_{ks}\cdots 0]^T) \\
&= B^{-1}(b_r + [0\cdots\delta a_{ks}\cdots 0]^T) \\
&= B^{-1}b_r + \overleftarrow{\beta}_k\ \delta a_{ks} \\
&= e_r + \overleftarrow{\beta}_k\ \delta a_{ks} \\
&\text{(Since } B^{-1}b_r = e_r \text{ being basic variable)}
\end{aligned}
$$

$$
\begin{aligned}
B^{-1}a_s' &= e_r + \overleftarrow{\beta}_k\ \delta a_{ks} = \lambda \text{ (say)} \\
&= (\lambda_1\ \lambda_2 \ldots\ldots \lambda_r \ldots\ldots \lambda_m)^T
\end{aligned}
$$

$$
\Rightarrow \begin{Bmatrix} \lambda_i &= \beta_{ik}\delta a_{ks}\ \forall\, i \neq r \\ \lambda_r &= 1 + \beta_{rk}\delta a_{ks} \end{Bmatrix} \qquad (27)
$$

$$
\begin{aligned}
B^{-1}B' &= B^{-1}(b_1\ b_2 \cdots b_r' \cdots b_m) \\
&= (e_1 \cdots e_{r-1}\quad \lambda \quad e_{r+1} \cdots e_m)
\end{aligned}
$$

It can be verified that if $\lambda_r \neq 0,$ then

$$
(B^{-1}B')^{-1} = (e_1 \cdots e_{r-1}\ \lambda\ e_{r+1} \cdots e_m)^{-1}
$$

$$
= \begin{pmatrix}
1 & 0\cdots & \lambda_1 & \cdots & 0 \\
0 & 1\cdots & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\vdots & & \lambda_r & 0 & \\
 & & \vdots & 1 & \\
\cdots & & \lambda_m & 0 & 1
\end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix}
1 & 0\cdots & -\lambda_1/\lambda_r & \cdots & 0 \\
0 & 1\cdots & -\lambda_2/\lambda_r & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
\vdots & & 1/\lambda_r & 0 & \\
 & & \vdots & 1 & \\
\cdots & & -\lambda_m/\lambda_r & 0 & 1
\end{pmatrix} = P
$$

$$
(B')^{-1}(B^{-1})^{-1} = P,
$$

$$
(B')^{-1} = PB^{-1}.
$$

So B' is non-singular iff $\lambda_r \neq 0.$

$$
X_B' = (B')^{-1}b - \sum_{a_j \in N_1} Y_j' l_j - \sum_{a_j \in N_2} Y_j' u_j
$$

$$
(B')^{-1}b = (PB^{-1})b
$$

$$
= \begin{pmatrix}
(B^{-1}b)_1 - \dfrac{\lambda_1}{\lambda_r}(B^{-1}b)_r \\
(B^{-1}b)_2 - \dfrac{\lambda_2}{\lambda_r}(B^{-1}b)_r \\
\vdots \\
\dfrac{(B^{-1}b)_r}{\lambda_r} \\
\vdots \\
(B^{-1}b)_m - \dfrac{\lambda_m}{\lambda_r}(B^{-1}b)_r
\end{pmatrix}, \qquad (28)
$$

Using (28) and (29),

$$
X_B' = \begin{pmatrix}
x_{B_1} - \dfrac{\lambda_1}{\lambda_r} x_{B_r} \\
x_{B_2} - \dfrac{\lambda_2}{\lambda_r} x_{B_r} \\
\vdots \\
\dfrac{x_{B_r}}{\lambda_r} \\
\vdots \\
x_{B_m} - \dfrac{\lambda_m}{\lambda_r} x_{B_r}
\end{pmatrix}.
$$

For solution X_B' to be feasible, $l_{B_i} \le x_{B_i}' \le u_{B_i}\ \forall\, i = 1,2,\cdots,m.$

$$
\left.\begin{aligned}
&\text{i.e. } l_{B_i} \le x_{B_i} - \frac{\lambda_i}{\lambda_r} \le u_{B_i}\quad \forall\ \ i \neq r \\
&\text{and}\quad l_{B_r} \le \frac{x_{B_r}}{\lambda_r} \le u_{B_r}
\end{aligned}\right\} (30)
$$

For solution to be feasible λ_r should be positive and also satisfy (30).Using (27) and (30)

$$
\text{i.e. } l_{B_i} \le x_{B_i} - \frac{\beta_{ik}\delta a_{ks}}{1 + \beta_{rk}\delta a_{ks}} x_{B_r} \le u_{B_i},
$$

$$
\Rightarrow \begin{Bmatrix}
l_{B_i}(1 + \beta_{rk}\delta a_{ks}) \le \\
x_{B_i}(1 + \beta_{rk}\delta a_{ks}) - \beta_{ik}\delta a_{ks} x_{B_r} \le \\
u_{B_i}(1 + \beta_{rk}\delta a_{ks})\ \forall\, i \neq r
\end{Bmatrix}
$$

and

$$\left\{\begin{array}{l} l_{B_r}(1+\beta_{rk}\delta a_{ks}) \le x_{B_r} \le \\ \qquad u_{B_r}(1+\beta_{rk}\delta a_{ks}) \end{array}\right\} \quad (31)$$

[using (1.27)]

$$\Rightarrow \left\{\begin{array}{l} l_{B_i} + l_{B_i}\beta_{rk}\delta a_{ks} \le \\ \qquad x_{B_i} + [x_{B_i}\beta_{rk} - \beta_{ik}x_{B_r}]\delta a_{ks} \le \\ \qquad\qquad u_{B_i} + \beta_{rk}u_{B_i}\delta a_{ks} \end{array}\right\} \quad (32)$$

Inequality (31) implies that

$$\Rightarrow \left\{\frac{x_{B_r}-l_{B_r}}{l_{B_r}\beta_{rk}} \mid l_{B_r}\beta_{rk} < 0\right\} \le \delta a_{ks} \le \left\{\frac{x_{B_r}-l_{B_r}}{l_{B_r}\beta_{rk}} \mid l_{B_r}\beta_{rk} > 0\right\} \quad (33)$$

and

$$\Rightarrow \left\{\frac{u_{B_r}-x_{B_r}}{-u_{B_r}\beta_{rk}} \mid u_{B_r}\beta_{rk} > 0\right\} \le \delta a_{ks} \le \left\{\frac{u_{B_r}-x_{B_r}}{-u_{B_r}\beta_{rk}} \mid u_{B_r}\beta_{rk} < 0\right\} \quad (34)$$

First part of inequality (32) implies that

$$\Rightarrow \max_{i,i\ne r}\left\{\frac{x_{B_i}-l_{B_i}}{\gamma_i} \mid \gamma_i < 0\right\} \le \delta a_{ks} \le \min_{i,i\ne r}\left\{\frac{x_{B_i}-l_{B_i}}{\gamma_i} \mid \gamma_i > 0\right\} \quad (35)$$

where $\gamma_i = \beta_{ik}x_{B_r} - \beta_{rk}(x_{B_i} - l_{B_i})$. (36)

And second part of inequality (32) implies that

$$\Rightarrow \max_{i,i\ne r}\left\{\frac{u_{B_i}-x_{B_i}}{\eta_i} \mid \eta_i > 0\right\} \le \delta a_{ks} \le \min_{i,i\ne r}\left\{\frac{u_{B_i}-x_{B_i}}{\eta_i} \mid \eta_i < 0\right\}, \quad (37)$$

where $\eta_i = \beta_{ik}x_{B_r} - \beta_{rk}(x_{B_i} - u_{B_i})$. (38)

(33), (34), (35) and (37) together give the range of δa_{ks} so that $X_B^{'}$ is a feasible solution of new problem. Note that, if for some i, $u_{B_i} = \infty$, then for finding the range of δa_{ks} for solution $X_B^{'}$ to be optimal, we do not consider the inequality (37) corresponding to that i.

Range of δa_{ks} for $X_B^{'}$ to be optimal solution

$$Z_j^{'} - C_j^{'} = Z_j^{'} - C_j = \sum_{i=1}^{m} C_{B_i} y_{ij}^{'} - C_j$$

$$= \sum_{i=1, i\ne r}^{m} C_{B_i}\left(y_{ij} - \frac{\lambda_i}{\lambda_r} y_{rj}\right) + C_{B_r}\frac{y_{rj}}{\lambda_r} - C_j$$

(using (28))

$$= \left[\begin{array}{l} \left(\sum_{i=1}^{m} C_{B_i} y_{ij} - C_j\right) - C_{B_r} y_{rj} + \\ C_{B_r}\frac{y_{rj}}{\lambda_r} - \frac{y_{rj}}{\lambda_r}\sum_{i=1, i\ne r}^{m} C_{B_i}\lambda_i \end{array}\right]$$

$$= \left[\begin{array}{l} (Z_j - C_j) - C_{B_r} y_{rj}\left(\frac{-1+\lambda_r}{\lambda_r}\right) - \\ \frac{y_{rj}}{\lambda_r}\sum_{i=1, i\ne r}^{m} C_{B_i}\lambda_i \end{array}\right]$$

$$= (Z_j - C_j) - \frac{y_{rj}}{1+\beta_{rk}\delta a_{ks}}\sum_{i=1}^{m} C_{B_i}\beta_{ik}$$

$$= (Z_j - C_j) - \frac{y_{rj}\delta a_{ks}}{1+\beta_{rk}\delta a_{ks}} C_B \overleftarrow{\beta}_k .$$

$X_B^{'}$ will be an optimal solution of new problem if

$$Z_j^{'} - C_j^{'} \ge 0 \; \forall \, a_j \in N_1 \text{ and}$$
$$Z_j^{'} - C_j^{'} \le 0 \; \forall \, a_j \in N_2,$$

$$(Z_j - C_j) - \frac{y_{rj}\delta a_{ks}}{1+\beta_{rk}\delta a_{ks}} C_B \overleftarrow{\beta}_k \ge 0 \quad \forall \; a_j \in N_1,$$

$$\text{i.e.} \left\{\begin{array}{l} (Z_j - C_j) + \\ \delta a_{ks}[\beta_{rk}(Z_j - C_j) - y_{rj}\, C_B \overleftarrow{\beta}_k] \end{array}\right\} \ge 0.$$

$$\Rightarrow \max_{a_j\in N_1}\left\{-\frac{(Z_j - C_j)}{\alpha_j} \mid \alpha_j > 0\right\} \le \delta a_{ks} \le \min_{a_j\in N_1}\left\{-\frac{(Z_j - C_j)}{\alpha_j} \mid \alpha_j < 0\right\}, \quad (39)$$

where $\alpha_j = \beta_{rk}(Z_j - C_j) - y_{rj}\, C_B \overleftarrow{\beta}_k$. (40)

For $a_j \in N_2$, $(Z_j - C_j) + \delta a_{ks}\alpha_j \le 0$ implies that,

$$\Rightarrow \max_{a_j \in N_2}\left\{-\frac{(Z_j - C_j)}{\alpha_j} \mid \alpha_j < 0\right\} \le \delta a_{ks} \le \min_{a_j \in N_2}\left\{-\frac{(Z_j - C_j)}{\alpha_j} \mid \alpha_j > 0\right\}. \quad (41)$$

Relations (38) and (40) together give the required range of δa_{ks} for solution $X_B^{'}$ to be optimal solution of new problem. And (38), (40) ,(33), (34), (35), (37) give the range of δa_{ks} for solution $X_B^{'}$ to be optimal as well as feasible solution of new problem.

Remark 3: Solution of the problem when some $a_s \in B$ undergoes change

For simplicity, let x_1 be the basic variable in the optimum basic vector X_B for (P). Suppose we have to modify first input-output coefficient say a_{11}, in the column vector a_1 of x_1 to $a_{11}^{'}$. The modified column vector of x_1 will be

$$a_1^{'} = (a_{11}^{'} \ a_{21} \cdots \ a_{i1} \cdots \ a_{m1})^T.$$

Let $x_1^{'}$ indicate the level of this activity corresponding to the new column vector $a_1^{'}$. The previous column vector a_1 is no longer a part of the problem and it should be eliminated. Physically, $x_1^{'}$ replaces x_1 in the optimal tableau. This is same as addition of a variable and deletion of another. Construct a new problem by augmenting the present original tableau with new variable $x_1^{'}$ with its column vector $a_1^{'}$ and cost coefficient C_1 and $l_1 \le x_1^{'} \le u_1$. Also change the cost coefficient of x_1 to $-M$ ($+M$ in minimization case), where M is a very large positive number. This leads to the problem, which we call $(P^{'})$. In this new problem x_1 has high negative cost, so serves as an artificial variable. We can set $x_1^{'}$ at its lower bound/upperbound and make changes in X_B and objective function value as discussed during the addition of new activity. While solving the problem there are three possibilities.

(i) Only optimality is hampered.

(ii) Only feasibility is hampered.

(iii) Optimality as well as feasibility both are hampered.

Case (iii) can be reduced to case (i) or (ii) by changing the level of $x_1^{'}$ as discussed during the addition of a variable.

Case (i) In this case apply simplex algorithm and proceed. If x_1 attains its lower bound which is zero now. As problem $(P^{'})$ is feasible and x_1 is non-basic with $x_1 = 0$. Let $(\bar{x}_1 \cdots \bar{x}_n, \bar{x}_1^{'})$ be optimal basic feasible solution (OBFS) of $(P^{'})$.

Then $(\bar{x}_2 \cdots \bar{x}_n, \bar{x}_1^{'})$ will be optimal basic feasible solution of the required problem say (P^*). Let optimal objective function value of $(P^{'})$ be $Z^{'}$ and

$$Z^{'} = \left\{ \begin{array}{c} C_B(B^{-1}b) - \sum_{a_j \in N_1} (Z_j - C_j) l_j \\ - \sum_{a_j \in N_2} (Z_j - C_j) u_j \end{array} \right\}.$$

Then optimal objective function value of the required problem, $Z^* = Z^{'} + (Z_1 - C_1) l_1 = Z^{'}$, since $l_1 = 0$.

If x_1 is still basic and can be replaced, then replace it and delete its column from the optimal table. Otherwise the required problem will be infeasible (Murty, 1976).

Case (ii) Apply dual simplex algorithm till dual optimality or primal infeasibility criterion is satisfied. If $(P^{'})$ is feasible, then proceed as in case (i), otherwise required problem is also infeasible.

4 Concluding Remarks

Other changes like change in cost vector C, change in requirement vector b, addition of a variable and addition of a constraint can be easily discussed on the same line as for these three cases. In the previous discussion, we have considered change only in one element of matrix A. But the method, discussed above can be used to solve the problem when we consider more than one change in an activity vector. Similarly, Change in a row vector (more than one element at a time) is equivalent to addition of one constraint and deletion of another. Similarly change in a column vector (more than one element at a time) is equivalent to addition of one variable and deletion of another.

5 Numerical Illustration

(P) Maximize $Z = 3x_1 + 5x_2 + 3x_3$

subject to $x_1 + 2x_2 + x_3 \le 19,$

$$2x_1 + 4x_2 + 3x_3 \le 33,$$

where $1 \le x_1 \le 5$, $2 \le x_2 \le 7$, $0 \le x_3 \le 1$.

Solution: Optimal feasible solution of the problem (P) is $X^o = (5, 23/4, 0)$ and $Z = 175/4$. Optimal table showing this solution is:

Table 1

$L^* \rightarrow$			u_1	b	l_3	b	l_5
$C_j \rightarrow$			3	5	3	0	0
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	5/2	0	0	$-1/2$	1	$-1/2$
5	a_2	23/4	1/2	1	3/4	0	1/4
$Z_j - C_j \rightarrow$			$-1/2$	0	3/4	0	5/4

Z=175/4

(L* denotes the level of the variable in the solution.)

<u>Deletion of a variable</u>

(i) Deletion of a non-basic variable.

Let in <u>table</u> 1, $x_1 = u_1$ be dropped, then

$$X_B' = X_B + Y_1 u_1 = \begin{pmatrix} 5/2 \\ 23/4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}(5) = \begin{pmatrix} 5/2 \\ 33/4 \end{pmatrix},$$

$$Z' = \frac{175}{4} - \frac{1}{2} \times 5 = \frac{165}{4}.$$

Here optimality is hampered as $x_2 = 33/4 > u_2$. As x_1 is non-basic so on deleting the column Y_1 in table 1, we have

$L \rightarrow$			b	l_3	b	l_5
$C_j \rightarrow$			5	3	0	0
C_B	B	X_B	y_2	y_3	y_4	y_5
0	a_4	5/2	0	$-1/2$	1	$-1/2$
5	a_2	33/4	1	3/4	0	1/4
$Z_j - C_j \rightarrow$			0	3/4	0	5/4

Z=165/4

As $x_2 = 33/4 > 7$, so it departs and x_3 enters the basis. Applying dual simplex algorithm repeatedly optimal solution of perturbed problem is given by

$L \rightarrow$			u_2	u_3	b	b
$C_j \rightarrow$			5	3	0	0
C_B	B	X_B	y_2	y_3	y_4	y_5
0	a_4	4	2	1	1	0
0	a_5	2	4	3	0	1
$Z_j - C_j \rightarrow$			-5	-3	0	0

Z=38

(ii) Deletion of a basic variable.

Let $a_2 = b_2 \in B$ be deleted. Here we consider that $x_2 \ge 0$, so it serves as an artificial variable. Also $C_2 \rightarrow -M$. From table 1, making changes in $Z_j - C_j$ and Z accordingly, we have following table

$L \rightarrow$		u_1	b	l_3	b	l_5	
$C_j \rightarrow$		3	-M	3	0	0	
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	5/2	0	0	-1/2	1	-1/2
-M	a_2	23/4	1/2	1	3/4	0	1/4
$Z_j - C_j \rightarrow$			-M/2-3	0	-3M/4-3	0	–M/4

$Z = (-23M + 60)/4$

As $Z_3 - C_3 < 0$, so x_3 undergoes change. Applying simplex algorithm

$L \rightarrow$		u_1	b	l_3	b	l_5	
$C_j \rightarrow$		3	-M	3	0	0	
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	3	0	0	-1/2	1	-1/2
-M	a_2	5	1/2	1	3/4	0	1/4
$Z_j - C_j \rightarrow$			-M/2-3	0	-3M/4-3	0	-M/4

Z=-5M+18

Again $Z_5 - C_5 < 0$, so x_5 undergoes change. Applying simplex algorithm again, we have

$L \rightarrow$		u_1	b	l_3	b	l_5	
$C_j \rightarrow$		3	-M	3	0	0	
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	13	1	2	1	1	0
0	a_5	20	2	4	3	0	1
$Z_j - C_j \rightarrow$			-3	M	-3	0	0

Z=18

So the solution in above table is optimal and x_2 in non-basic and at lower bound, so on deleting Y_2, optimal solution of the perturbed problem is $Z^{'} = 18$ and $X^{'} = (5,1)$.

<u>Deletion of a constraint</u>

(i) Deletion of an inactive constraint.

As $x_4 = 5/2 > 0$, so first constraint is inactive. So to find the optimal solution of the perturbed problem, we delete the column Y_4 and first row from table 1 and there will be no change in X_B, Z and $Z_j - C_j$.

(ii) Deletion of an active constraint.

In problem (P), second constraint is active. To make it inactive, change $C_5 (= 0) \rightarrow M$ and calculate $Z_5^{'} - C_5^{'}$ in Table 1.

$L \rightarrow$		u_1	b	l_3	b	l_5	
$C_j \rightarrow$		3	5	3	0	M	
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	5/2	0	0	-1/2	1	-1/2
5	a_2	23/4	1/2	1	3/4	0	1/4
$Z_j - C_j \rightarrow$			-1/2	0	3/4	0	5/4-M

Z=175/4.

As $Z_5^{'} - C_5^{'} < 0$, so x_5 undergoes change.

Applying simplex algorithm repeatedly, we have

$L \rightarrow$		l_1	l_2	l_3	b	b	
$C_j \rightarrow$		3	5	3	0	M	
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	14	1	2	1	1	0
M	a_5	23	2	4	3	0	1

$Z_j - C_j \rightarrow$	2M-3	4M-5	3M-3	0	0

Z=13+23M

Note that $x_5 > 0$, so second constraint is inactive. On deleting Y_5 and second row in the above table and also making changes in Z and $Z_j - C_j$, we have the following table

		L $\rightarrow$	l_1	l_2	l_3	b
		$C_j \rightarrow$	3	5	3	0
C_B	B	X_B	y_1	y_2	y_3	y_4
0	a_4	14	1	2	1	1
$Z_j - C_j \rightarrow$			-3	-5	-3	0

Z=13

This solution in above table is feasible but not optimal, so applying simplex algorithm repeatedly; the optimal solution of the perturbed problem is given by

		L $\rightarrow$	u_1	b	u_3	l_4
		$C_j \rightarrow$	3	5	3	0
C_B	B	X_B	y_1	y_2	y_3	y_4
0	a_2	6	1/2	1	1/2	1/2
Z=51, $Z_j - C_j \rightarrow$			-1/2	0	-1/2	5/2

<u>Change in A</u>

(i) Let $a_1 \notin B$ undergoes change and $a_{11} \rightarrow a_{11} + \delta a_{11}$. Here $s = 1, k = 1$. When a_{11} changes then X_B, Z and only $Z_1 - C_1$ undergo change as discussed earlier.

$$B^{-1} = \begin{pmatrix} 1 & -1/2 \\ 0 & 1/4 \end{pmatrix}, \; C_B = (0 \;\; 5).$$

$$\Rightarrow C_B \overleftarrow{\beta}_1 = 0.$$

Range of δa_{11} for maintaining optimality is given by

$$\left\{ -\frac{(Z_1 - C_1)}{C_B \overleftarrow{\beta}_1} \,\middle|\, C_B \overleftarrow{\beta}_1 > 0 \right\} \le \delta a_{11} \le \left\{ -\frac{(Z_1 - C_1)}{C_B \overleftarrow{\beta}_1} \,\middle|\, C_B \overleftarrow{\beta}_1 < 0 \right\}.$$

Since $C_B \overleftarrow{\beta}_1 = 0$, so new solution will be optimal for all values of δa_{11}.

Range of δa_{11} for maintaining feasibility is given by

$$\max_i \left\{ \begin{matrix} \left(\dfrac{x_{B_i} - l_{B_i}}{\beta_{i1} u_1} \,\middle|\, \beta_{i1} u_1 < 0 \right), \\ \left(\dfrac{u_{B_i} - x_{B_i}}{-\beta_{i1} u_1} \,\middle|\, \beta_{i1} u_1 > 0 \right) \end{matrix} \right\} \le \delta a_{11} \le \min_i \left\{ \begin{matrix} \left(\dfrac{u_{B_i} - x_{B_i}}{-\beta_{i1} u_1} \,\middle|\, \beta_{i1} u_1 < 0 \right), \\ \left(\dfrac{x_{B_i} - l_{B_i}}{\beta_{i1} u_1} \,\middle|\, \beta_{i1} u_1 > 0 \right) \end{matrix} \right\}.$$

$\beta_{11} u_1 = 5 > 0, \; \beta_{21} u_1 = 0$.

For $i = 1$, we have $-\infty \le \delta a_{11} \le 1/2$, and for $i = 2$, solution is feasible for all values of δa_{11}. Thus range of δa_{11} for maintaining optimality as well as feasibility is

$$-\infty \le \delta a_{11} \le 1/2.$$

Let $\delta a_{11} = 3$, then feasibility will be hampered.

$$X_B' = \begin{pmatrix} 5/2 \\ 23/4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} 5 = \begin{pmatrix} -1/2 \\ 23/4 \end{pmatrix}, \; Z' = 175/4.$$

$$Z_1' - C_1' = (Z_1 - C_1) + 0 = -1/2,$$

$$Y_1^{'} = Y_1 + \overleftarrow{\beta}_1 \delta a_{11} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} 3 = \begin{pmatrix} 3 \\ 1/2 \end{pmatrix}.$$

L →	u_1	b	l_3	b	l_5	
C_j →	3	5	3	0	0	

C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
0	a_4	-1/2	3	0	-1/2	1	-1/2
5	a_2	23/4	1/2	1	3/4	0	1/4
$Z_j - C_j$ →			-1/2	0	3/4	0	5/4

$Z = 175/4$

As $x_4 < 0$, so it departs. Applying dual simplex method, OBFS of perturbed problem will be given by

L →	b	b	l_3	l_4	l_5
C_j →	3	5	3	0	0

C_B	B	X_B	y_1	y_2	y_3	y_4	y_5
3	a_1	29/6	1	0	-1/6	1/3	-1/6
5	a_2	35/6	0	1	5/6	-1/6	1/3
$Z_j - C_j$ →			0	0	2/3	1/6	7/6

Z=131/3

(ii) Let $a_2 = b_2 \in B$ undergoes change and $a_{22} \to a_{22} + \delta a_{22}$. Here $r = 2, k = 2$. When δa_{22} undergoes change, let $X_B \to X_B^{'}$.

Range of δa_{22} for solution $X_B^{'}$ to be optimal solution of the perturbed problem, for $i = 2, r = 2,\ l_{B_2}\beta_{22} = 1/2 > 0$, $u_{B_2}\beta_{22} = 7/4 > 0.$

$$\Rightarrow \left\{ \frac{x_{B_2} - l_{B_2}}{l_{B_2}\beta_{22}} \,|\, l_{B_2}\beta_{22} < 0 \right\} \le \delta a_{22} \le \left\{ \frac{x_{B_2} - l_{B_2}}{l_{B_2}\beta_{22}} \,|\, l_{B_2}\beta_{22} > 0 \right\}$$ and

$$\Rightarrow \left\{ \frac{u_{B_2} - x_{B_2}}{-u_{B_2}\beta_{22}} \,|\, u_{B_2}\beta_{22} > 0 \right\} \le \delta a_{22} \le \left\{ \frac{u_{B_2} - x_{B_2}}{-u_{B_2}\beta_{22}} \,|\, u_{B_2}\beta_{22} < 0 \right\}$$

$$\Rightarrow -\frac{5}{7} \le \delta a_{22} \le \frac{15}{2}. \qquad (42)$$

For $i = 1$,

$$\gamma_1 = \beta_{12} x_{B_2} - \beta_{22} - \beta_{22}(x_{B_1} - l_{B_1}) = -\frac{7}{2} < 0,$$

Note that inequality (37) is not used here as $u_{B_1} = \infty.$

$$\Rightarrow -\frac{5}{7} \le \delta a_{22}. \qquad (43)$$

Combining (41) and (42), range of δa_{22} for solution $X_B^{'}$ to be feasible is

$$\Rightarrow -\frac{5}{7} \le \delta a_{22} \le \frac{15}{2}. \qquad (44)$$

Range of δa_{22} to be optimal:

$$\alpha_j = \beta_{22}(Z_j - C_j) - y_{2j}(C_B \overleftarrow{\beta}_2),$$

$$C_B \overleftarrow{\beta}_2 = \frac{5}{4}.$$

$$N_1 = \{a_3, a_5\},\ N_2 = \{a_1\}.$$

$$\alpha_1 = \beta_{22}(Z_1 - C_1) - y_{21}(C_B \overleftarrow{\beta}_2) = -\frac{1}{8},$$

$$\alpha_3 = \beta_{22}(Z_3 - C_3) - y_{23}(C_B \overleftarrow{\beta}_2) = -\frac{3}{4},$$

and $\alpha_5 = \beta_{22}(Z_5 - C_5) - y_{25}(C_B \overleftarrow{\beta}_2) = 0.$

$$\max\left\{\begin{array}{l}\left(-\frac{(Z_j-C_j)}{\alpha_j}\,|\,\alpha_j>0, a_j\in N_1\right),\\ \left(-\frac{(Z_j-C_j)}{\alpha_j}\,|\,\alpha_j<0, a_j\in N_2\right)\end{array}\right\}$$
$$\leq \delta a_{rk} \leq$$
$$\min\left\{\begin{array}{l}\left(-\frac{(Z_j-C_j)}{\alpha_j}\,|\,\alpha_j<0,\ a_j\in N_1\right),\\ \left(-\frac{(Z_j-C_j)}{\alpha_j}\,|\,\alpha_j>0, a_j\in N_2\right)\end{array}\right\}.$$

Thus $X_B^{'}$ will be optimal if

$$-4 \leq \delta a_{22} \leq 1 \qquad (45) \qquad (2)$$

From (43) and (44), range of δa_{22} for solution $X_B^{'}$ to be OBFS of the perturbed problem is

$$-\frac{5}{7} \leq \delta a_{22} \leq 1.$$

Let $\delta a_{22} = 2$. For solving the problem, make $x_2 \geq 0$ and introduce a new variable $x_2^{'}$ with $C_2 = 5$ and $2 \leq x_2^{'} \leq 7$.

Set $x_2^{'} = l_2^{'} = 2,$ then $Z_2^{'} - C_2^{'} = 5/2$ and $X_B^{'} = X_B - Y_2^{'} l_2$.

$$a_2^{'} = \begin{pmatrix}2\\6\end{pmatrix},\ Y_2^{'} = \begin{pmatrix}1 & -1/2\\ 0 & 1/4\end{pmatrix}\begin{pmatrix}2\\6\end{pmatrix} = \begin{pmatrix}-1\\3/2\end{pmatrix}.$$

$$\Rightarrow X_B^{'} = \begin{pmatrix}5/2\\23/4\end{pmatrix} - \begin{pmatrix}-1\\3/2\end{pmatrix}(2) = \begin{pmatrix}9/2\\11/4\end{pmatrix}.$$

$$Z^{'} = Z - (Z_2^{'} - C_2^{'})l_2^{'} = \frac{175}{4} - \frac{5}{2}\times 2 = \frac{155}{4}.$$

From table 1,

$L \rightarrow$			u_1	b	l_3	b	l_5	$l_2^{'}$
$C_j \rightarrow$			3	5	3	0	0	5
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5	$y_2^{'}$
0	a_4	9/2	0	0	-1/2	1	-1/2	-1
5	a_2	11/4	1/2	1	3/4	0	1/4	3/2
$Z_j - C_j \rightarrow$			-1/2	0	3/4	0	5/4	5/2

Z=155/4

This solution is optimal as well as feasible. Now we delete x_2.

On changing $C_2 = 5 \rightarrow -M$ and calculate $Z, Z_j - C_j, \forall\, a_j,$ in above table

$L \rightarrow$			u_1	b	l_3	b	l_5	$l_2^{'}$
$C_j \rightarrow$			3	-M	3	0	0	5
C_B	B	X_B	y_1	y_2	y_3	y_4	y_5	$y_2^{'}$
0	a_4	9/2	0	0	-1/2	1	-1/2	-1
-M	a_2	11/4	1/2	1	3/4	0	1/4	3/2

$Z_j - C_j \rightarrow$ -M/2-3 0 -3M/4-3 0 -M/4, –3M/2-5

Z=25-11M/4

As $Z_2^{'} - C_2^{'} < 0$, so $x_2^{'}$ undergoes change. Applying simplex algorithm repeatedly, we get the following table

$L \rightarrow$	u_1	$l_2 = 0$	u_3	b	l_5	b
$C_j \rightarrow$	3	-M	3	0	0	5

C_B	B	X_B	y_1	y_2	y_3	y_4	y_5	y_2'
0	a_4	19/3	1/3	2/3	0	1	-1/3	0
5	a_2'	10/3	1/3	2/3	1/2	0	1/6	1
$Z_j - C_j \rightarrow$			-4/3	10/3+M	-1/2	0	5/6	0

Z=104/3

Now x_2 is non-basis and at lower bound, so delete Y_2. There will be no change in Z and X_B. So the optimal solution of perturbed problem will be $X_B = (5, 10/3, 1)$ and $Z = 104/3$.

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